

Unit 4: Functions Involving Parameters, Vectors, and Matrices

Unit 4 introduces three tools that describe motion and structure: parametric equations, which track a moving point with a parameter; vectors, which carry a direction and a size; and matrices, which organize numbers so systems of equations can be solved in bulk. One honest note up front: this unit is not assessed on the AP Precalculus exam. It is still part of the course, and parametric equations, vectors, and matrices all return in calculus, where they do real work.

AP Precalculus

Parameters, Vectors, and Matrices

About 11 minutes to read

How to use this guide

Read it in order the first time. Parametric equations give you a way to describe motion, vectors give you direction and size in the plane, and matrices give you a compact way to handle systems of equations. Every worked example ends with a verification step. Cover the verification, do the arithmetic yourself, then uncover and compare. That habit is the whole point of this unit, because each computation is only as good as its check.

After the first read, use the trap boxes and the confusion table to review the mistakes this material actually produces. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. This unit is not assessed on the AP Precalculus exam, so it earns no test-day points. That is a College Board decision, not a judgment on the material. Parametric equations, vectors, and matrices are standard tools in calculus and in the sciences that use it, so study this unit for your course grade and for calculus readiness. The practice questions below are written for course mastery, not for the exam.

Parametric Equations

A **parametric equation** describes a curve by giving x and y each as a function of a third variable, the **parameter**, usually called t . Instead of y as a function of x , you get a pair: $x = f(t)$, $y = g(t)$. As t varies, the point $(x(t), y(t))$ moves and traces out the curve, with t telling you where the point is at each moment. Time is the most common interpretation of t , which is why parametrics are the natural language for motion.

Take $x = 2t + 1$, $y = t^2 - 3$. Plugging in values of t gives points on the curve:

T	X = 2T + 1	Y = T ² - 3	POINT
0	1	-3	(1, -3)
1	3	-2	(3, -2)
2	5	1	(5, 1)
3	7	6	(7, 6)

Eliminating the parameter means solving one equation for t and substituting into the other, which converts the parametric pair into a single Cartesian equation. From $x = 2t + 1$, $t = (x - 1)/2$. Substituting into y gives $y = ((x - 1)/2)^2 - 3$, a parabola. Verify with the $t = 2$ row of the table: at $x = 5$, $y = ((5 - 1)/2)^2 - 3 = (2)^2 - 3 = 4 - 3 = 1$, which matches the point $(5, 1)$. The elimination checks out.

The parametric form carries information the Cartesian form does not: the **direction of motion** and the speed. The table shows the point moving right and up as t increases, but $y = ((x - 1)/2)^2 - 3$ alone cannot tell you that. Projectile motion is the classic application: $x(t)$ tracks horizontal position, $y(t)$ tracks height, and t is time.

Trap. Eliminating the parameter can silently widen the domain. If t is restricted, say $t \geq 0$, then $x = 2t + 1$ only reaches $x \geq 1$, and the full parabola $y = ((x - 1)/2)^2 - 3$ includes points the motion never visits. Keep the restriction on t , or state the restricted x -values alongside the Cartesian equation. Direction of motion is lost the same way: the graph shows the path, not which way the point travels along it.

Vectors

A **vector** is a quantity with a direction and a size. In the plane, a vector is written in **component form** as $v = \langle x, y \rangle$, where x is the horizontal component and y is the vertical component. The **magnitude** (or length) of $v = \langle x, y \rangle$ is $|v| = \sqrt{x^2 + y^2}$, which is just the Pythagorean theorem applied to the components. For $v = \langle 3, 4 \rangle$, $|v| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. The 3-4-5 triangle makes this one easy to check.

Vector addition is done component by component: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$. Geometrically, place the tail of the second vector at the tip of the first and read the result from the free tail to the free tip. For example, $\langle 3, 4 \rangle + \langle 1, -2 \rangle = \langle 4, 2 \rangle$. Check the lengths: the sum has magnitude $\sqrt{16 + 4} = \sqrt{20} \approx 4.47$, which is less than $5 + \sqrt{5} \approx 7.24$, the sum of the two magnitudes. That is expected, and it is the point of the trap below.

Scalar multiplication stretches or shrinks a vector: $k\langle x, y \rangle = \langle kx, ky \rangle$. A negative k also reverses the direction. For $2\langle 3, 4 \rangle = \langle 6, 8 \rangle$, the magnitude is $\sqrt{36 + 64} = \sqrt{100} = 10$, exactly twice the original magnitude 5. Doubling every component doubles the length, which is a good quick check whenever you scale a vector.

A **unit vector** has magnitude 1 and records a pure direction. To make a unit vector from v , divide by its magnitude: $v/|v|$. For $v = \langle 3, 4 \rangle$, the unit vector is $\langle 3/5, 4/5 \rangle$, and checking gives $\sqrt{(3/5)^2 + (4/5)^2} = \sqrt{9/25 + 16/25} = \sqrt{1} = 1$.

Trap. The magnitude of a sum is not the sum of the magnitudes. In the example above, $|\langle 3, 4 \rangle + \langle 1, -2 \rangle| = \sqrt{20} \approx 4.47$, but $|\langle 3, 4 \rangle| + |\langle 1, -2 \rangle| = 5 + \sqrt{5} \approx 7.24$. Lengths add only when the vectors point in exactly the same direction. Component addition is the reliable route; never add the lengths and call it the length of the sum.

The Dot Product

The **dot product** combines two vectors into a single number, a scalar. For $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$, the dot product is $u \cdot v = u_1v_1 + u_2v_2$. Multiply matching components, then add. The result is not a vector, and that single fact is the most tested confusion in this section.

Worked example: $u = \langle 1, 2 \rangle$, $v = \langle 3, -1 \rangle$. The dot product is $(1)(3) + (2)(-1) = 3 - 2 = 1$. The magnitudes are $|u| = \sqrt{1 + 4} = \sqrt{5}$ and $|v| = \sqrt{9 + 1} = \sqrt{10}$.

The dot product also measures the **angle between vectors** through the formula $u \cdot v = |u| |v| \cos \theta$, so $\cos \theta = (u \cdot v) / (|u| |v|)$. For the example, $\cos \theta = 1 / (\sqrt{5} \cdot \sqrt{10}) = 1/\sqrt{50}$. Since $\sqrt{50} = 5\sqrt{2}$, this is $1/(5\sqrt{2})$, and rationalizing gives $\sqrt{2}/10$. Verify the simplification: $1/(5\sqrt{2})$ multiplied top and bottom by $\sqrt{2}$ is $\sqrt{2}/(5 \cdot 2) = \sqrt{2}/10$. Then $\theta = \arccos(\sqrt{2}/10) \approx 81.9^\circ$, because $\cos(81.9^\circ) \approx 0.1414$ and $\sqrt{2}/10 \approx 0.1414$. The arithmetic closes.

A dot product of zero has a clean geometric meaning. For nonzero vectors, $u \cdot v = 0$ exactly when the vectors are **perpendicular**, since $\cos 90^\circ = 0$. Check: $\langle 3, -2 \rangle \cdot \langle 2, 3 \rangle = (3)(2) + (-2)(3) = 6 - 6 = 0$, so those two vectors meet at a right angle. This zero test is the fastest way to check perpendicularity on paper.

Trap. The dot product is a scalar, not a vector. Writing $u \cdot v = \langle \text{something}, \text{something} \rangle$ is always wrong. The angle brackets belong to vectors; the dot product belongs to the number line. If your answer to a dot product has components, recompute.

Matrices: Addition and Scalar Multiplication

A **matrix** is a rectangular array of numbers. Its **dimensions** (also called its order) are given as rows $\times$ columns, so a matrix with 2 rows and 3 columns is a 2×3 matrix. Dimensions control everything in this section: they decide which operations are even allowed.

Matrix addition adds matching entries, and it requires both matrices to have the same dimensions. Entry by entry:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Scalar multiplication multiplies every entry by the scalar: $3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$. Like vector scalar multiplication, this is an entry-by-entry operation with no interaction between positions.

These two operations behave the way you would hope: matrix addition is commutative and associative, and scalar multiplication distributes over addition. The arithmetic is simple. The discipline is checking dimensions first, because an operation on mismatched matrices is not hard, it is undefined.

Trap. Check dimensions before you touch the numbers. Adding a 2×3 matrix to a 2×2 matrix is undefined, not zero and not a partial sum. Write the dimensions of both matrices first, then decide whether the operation exists. This single habit prevents most of the errors in matrix arithmetic.

Matrix Multiplication

Matrix multiplication is defined only when the number of columns in the first matrix equals the number of rows in the second. Multiplying an $m \times n$ matrix by an $n \times p$ matrix gives an $m \times p$ result: the inner dimensions must match, and the outer dimensions give the shape of the answer.

Each entry of the product is a dot product of a row of the first matrix with a column of the second. Entry (i, j) uses row i from the left matrix and column j from the right matrix. Full 2×2 example, with every entry verified:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

- Entry $(1, 1)$: row 1 $\cdot$ column 1 = $(1)(5) + (2)(7) = 5 + 14 = 19$
- Entry $(1, 2)$: row 1 $\cdot$ column 2 = $(1)(6) + (2)(8) = 6 + 16 = 22$
- Entry $(2, 1)$: row 2 $\cdot$ column 1 = $(3)(5) + (4)(7) = 15 + 28 = 43$
- Entry $(2, 2)$: row 2 $\cdot$ column 2 = $(3)(6) + (4)(8) = 18 + 32 = 50$

So the product is $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. Notice that entry $(1, 1)$ is not $1 \cdot 5 = 5$. Entry-by-entry multiplication is a different, much rarer operation; matrix multiplication always pairs a row with a column.

Order matters. In general $AB \neq BA$, and one product can exist while the other does not. A 2×3 matrix times a 3×2 matrix gives a 2×2 result, but reversing the order gives a 3×3 . Even for square matrices the two orders usually disagree, as the practice questions show.

Trap. Multiplying in the wrong order or with mismatched dimensions is the standard matrix error. Before computing, write the dimensions as $(m \times n)(n \times p)$ and confirm the inner numbers match; the product will be $m \times p$. If the inner numbers differ, stop. The operation does not exist, and no amount of arithmetic fixes that.

Determinants and Inverse Matrices

The **determinant** of a 2×2 matrix is a single number computed as $ad - bc$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: multiply down the main diagonal, subtract the product down the other diagonal. For the product example from the last page, $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$. The sign survives: determinants can be negative, and a negative determinant is a valid answer, not a signal to recompute.

The determinant decides whether a matrix has an **inverse**. The inverse of A , written A^{-1} , is the matrix that undoes A : $A \times A^{-1} = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the **identity matrix**. For a 2×2 matrix with $\det \neq 0$, the formula is $A^{-1} = (1/\det) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$: swap a and d , negate b and c , and divide everything by the determinant. If $\det = 0$, no inverse exists, and the formula is meaningless because it divides by zero.

Worked example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The determinant is $(2)(1) - (1)(1) = 2 - 1 = 1$. The inverse is $(1/1) \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Verify by multiplying $A \times A^{-1}$ entry by entry:

- Entry (1, 1): $(2)(1) + (1)(-1) = 2 - 1 = 1$
- Entry (1, 2): $(2)(-1) + (1)(2) = -2 + 2 = 0$
- Entry (2, 1): $(1)(1) + (1)(-1) = 1 - 1 = 0$
- Entry (2, 2): $(1)(-1) + (1)(2) = -1 + 2 = 1$

The result is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, so the inverse is correct. This check is the reason the formula is trustworthy: whenever you compute an inverse, multiply it back and confirm you get the identity.

Trap. The inverse formula has two sign moves, and getting one wrong is the classic error. Swap a and d (positions stay positive), then negate b and c . A frequent wrong answer is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ for the example above, which negates without swapping. Write the formula first, then substitute, then verify with $A \times A^{-1} = I$. The verification catches every version of this mistake.

Solving Systems with Matrices

A system of linear equations can be written as a single matrix equation $AX = B$. The matrix A holds the coefficients, X holds the variables, and B holds the constants. If A has an inverse, multiplying both sides on the left by A^{-1} gives $X = A^{-1}B$, which is the solution.

Example: $2x + y = 5$ and $x + y = 3$. Here $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. From the last page, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, so $X = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Computing: $x = (1)(5) + (-1)(3) = 5 - 3 = 2$, and $y = (-1)(5) + (2)(3) = -5 + 6 = 1$. Check in the original equations: $2(2) + 1 = 5$ and $2 + 1 = 3$. Both hold, so $(2, 1)$ is the solution. This is the payoff of the whole unit: one matrix inverse solves any system with the same coefficients.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Dot product (scalar) vs vector result	The dot product $u \cdot v$ is a single number. If your answer has angle brackets, you computed something else. Vectors have components; dot products do not.
Matrix multiplication order	AB and BA are different operations and usually give different results. Write the dimensions first and never swap the order to make the arithmetic easier.
Determinant $ad - bc$ vs $bc - ad$	Main diagonal first, then subtract the other diagonal. Reversing the subtraction flips the sign, and the inverse formula inherits the error through the $1/\det$ factor.
Inverse formula: swap vs negate	Swap a and d , negate b and c . Negating without swapping, or swapping without negating, both give a matrix that fails the $A \times A^{-1} = I$ check.
Parametric direction vs Cartesian graph	The Cartesian equation shows the path; the parameter shows the direction and speed along it. Eliminating t also drops any domain restriction t carried, so state restricted ranges explicitly.

Practice Questions

Original questions written for this guide. This unit is not on the AP exam, so these test course mastery, not exam technique. Answers and explanations are on the next page, so complete the questions before checking them.

1. The parametric equations $x = 2t + 1$ and $y = t^2 - 3$ describe a moving point. At $t = 2$, the point is at

1. (1, 5)
2. (5, 1)
3. (4, 1)
4. (3, -2)

2. The magnitude of the vector $v = \langle -6, 8 \rangle$ is

1. 14
2. 100
3. 10
4. 2

3. For $u = \langle 2, 5 \rangle$ and $v = \langle -1, 4 \rangle$, the dot product $u \cdot v$ equals

1. 18
2. 22
3. -18
4. 20

4. For $u = \langle 3, -2 \rangle$ and $v = \langle 2, 3 \rangle$, the angle θ between the vectors is

1. 0°
2. 45°
3. 180°
4. 90°

5. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ equals

1. $\begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix}$
2. $\begin{bmatrix} 6 & 8 \\ 10 & 11 \end{bmatrix}$
3. $\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

6. Let $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Which of the following is true?

1. $PQ \neq QP$, so the order of multiplication matters
2. $PQ = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$
3. $QP = \begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix}$
4. $PQ = QP = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

7. The determinant of $\begin{bmatrix} 4 & 7 \\ 2 & 5 \end{bmatrix}$ is

1. -6
2. 6
3. 34
4. 18

8. For $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, the inverse matrix A^{-1} is

1. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}$
4. $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

Answer Key

- 1. B.** At $t = 2$, $x = 2(2) + 1 = 5$ and $y = 2^2 - 3 = 1$, so the point is $(5, 1)$. A swaps the coordinates. C computes $x = 2t = 4$, dropping the $+1$. D substitutes $t = 1$ instead of $t = 2$.
- 2. C.** $|v| = \sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$. A adds the components $(-6 + 8 = 2)$, then absolute mishandled as 14). B squares but forgets the square root. D adds the components directly, which is not a magnitude.
- 3. A.** $u \cdot v = (2)(-1) + (5)(4) = -2 + 20 = 18$. B drops the negative sign on -1 , computing $2 + 20 = 22$. C negates the whole correct sum. D computes only $(5)(4) = 20$, losing the first term.
- 4. D.** $u \cdot v = (3)(2) + (-2)(3) = 6 - 6 = 0$, and a zero dot product means the vectors are perpendicular, so $\theta = 90^\circ$. A confuses a zero dot product with parallel vectors. B and C are guesses that ignore the dot product entirely.
- 5. C.** Addition is entry by entry: $[[1 + 5, 2 + 6], [3 + 7, 4 + 8]] = [[6, 8], [10, 12]]$. A multiplies the entries instead of adding. B adds correctly except the $(2, 2)$ entry. D doubles the first matrix instead of adding the second.
- 6. A.** $PQ = [[3, 1], [6, 4]]$ and $QP = [[5, 1], [4, 2]]$, so the two products differ and order matters. B reports QP's entries as PQ. C reports PQ's entries as QP. D confuses these nonzero products with the zero matrix.
- 7. B.** $\det = ad - bc = (4)(5) - (7)(2) = 20 - 14 = 6$. A reverses the subtraction, computing $bc - ad = -6$. C adds the products instead of subtracting, $20 + 14 = 34$. D multiplies down the columns, $(4)(7) - (2)(5) = 28 - 10 = 18$.
- 8. D.** $\det A = (3)(1) - (2)(1) = 1$, so $A^{-1} = (1/1)[[1, -2], [-1, 3]] = [[1, -2], [-1, 3]]$. Multiplying back gives $A \times A^{-1} = [[1, 0], [0, 1]]$, which confirms it. A keeps the original positions without swapping or negating. B negates b and c but does not swap a and d . C negates every entry instead of following the formula.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. This unit is not on the exam, so the goal here is genuine command of the tools, not test-day recall.

- Explain what the parameter t does in the pair $x = f(t)$, $y = g(t)$.
- Eliminate the parameter from $x = 2t + 1$, $y = t^2 - 3$ and verify the result at $t = 2$.
- Explain why eliminating a parameter can change the domain and lose the direction of motion.
- State the magnitude formula and compute the magnitude of $\langle 3, 4 \rangle$.
- Add two vectors and multiply a vector by a scalar, component by component.
- Compute a dot product and explain why the answer is a scalar, not a vector.
- Use $\cos \theta = (u \cdot v) / (|u| |v|)$ for $u = \langle 1, 2 \rangle$, $v = \langle 3, -1 \rangle$ and give $\theta \approx 81.9^\circ$.
- State the dimension rule for matrix multiplication: when it is defined and what size the product is.
- Multiply two 2×2 matrices by hand, pairing each row with each column.
- Compute the determinant of a 2×2 matrix as $ad - bc$.
- Write the 2×2 inverse formula and verify an inverse with $A \times A^{-1} = I$.
- Solve a 2×2 linear system using $X = A^{-1}B$ and check the solution.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Parameters, Vectors, and Matrices deck under AP Precalculus. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a course test date, add it in the Test Planner.

Key terms for this unit

Parameter, Parametric equations, Eliminating the parameter, Direction of motion, Vector, Component form, Magnitude, Scalar multiplication, Unit vector, Dot product, Angle between vectors, Perpendicular vectors, Matrix, Dimensions of a matrix, Matrix addition, Matrix multiplication, Determinant, Inverse of a matrix, Identity matrix, System of linear equations.

About this guide. Written for Rycal and aligned to the College Board AP Precalculus course framework, Unit 4. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.