

Unit 3: Trigonometric and Polar Functions

Unit 3 is about periodic behavior. It covers radian measure, arc length and sector area, the unit circle, the graphs of sine, cosine, and tangent and their transformations, inverse trigonometric functions, trigonometric identities, and polar coordinates and polar graphs.

AP Precalculus

Trigonometric and Polar Functions

About 14 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. Radian measure makes the unit circle usable, the unit circle gives you exact values, those values anchor the graphs, the graphs motivate the inverse functions, and identities and polar coordinates extend the whole system. Exam questions usually pair two of these ideas, for example a unit-circle value inside a transformed graph.

After the first read, use the trap boxes and the Confusions That Cost Points table to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Trigonometric and Polar Functions is about 30-35% of the AP Precalculus exam, the largest share of any single unit. The conversion factors, the unit-circle table, and the sinusoidal form $y = A \sin(B(x - C)) + D$ are the three things this unit keeps testing in different disguises.

3.1 Radian Measure

A **radian** is defined by the radius itself. One radian is the central angle that cuts off an arc whose length equals the radius. Because the full circumference is $2\pi r$, a full rotation is 2π radians, which means **$360^\circ = 2\pi$ radians** and therefore **$180^\circ = \pi$ radians**. A single degree is $\pi/180$ of a radian.

To convert, multiply by the right fraction. Degrees to radians: multiply by $\pi/180$. Radians to degrees: multiply by $180/\pi$. Work 225° into radians: $225 \times \pi/180 = 225\pi/180$. Both numbers share the factor 45, so $225\pi/180 = 5\pi/4$. Check it back: $5\pi/4$ means $5/4$ of 180° , and $5/4 \times 180 = 225$. So $225^\circ = 5\pi/4$ radians.

Now go the other way. Convert $11\pi/6$ radians to degrees: $11\pi/6 \times 180/\pi$. The π cancels, leaving $11 \times 180/6 = 11 \times 30 = 330^\circ$. Check it back: $330 \times \pi/180 = 330\pi/180 = 11\pi/6$ after dividing top and bottom by 30. The two directions undo each other exactly.

Trap. A calculator left in degree mode silently ruins every radian problem in this unit. Before any numeric trig computation, confirm the mode shows radians. An answer like $\sin(\pi/6) = -0.175$ means the calculator read $\pi/6$ as 0.52 degrees, not 30 degrees.

3.2 Arc Length and Sector Area

Two formulas run this section, and both require the angle in **radians**. For a circle of radius r and a central angle θ , the **arc length** is $s = r\theta$, and the **area of the sector** is $A = (1/2)r^2\theta$. Both come from taking a fraction $\theta/(2\pi)$ of the full circumference $2\pi r$ or the full area πr^2 .

Work an arc length. Take $r = 6$ and $\theta = 2\pi/3$. Then $s = r\theta = 6 \times 2\pi/3 = 12\pi/3 = 4\pi$. A quick sense check: $2\pi/3$ is one third of a full rotation, and one third of the circumference $2\pi(6) = 12\pi$ is 4π . The formula agrees.

Work a sector area with the same numbers. Take $r = 6$ and $\theta = 2\pi/3$. Then $A = (1/2)r^2\theta = (1/2)(36)(2\pi/3) = 18 \times 2\pi/3 = 36\pi/3 = 12\pi$. Check it the same way: one third of the full area $\pi(6)^2 = 36\pi$ is 12π . Both formulas are just the radian fraction of the whole.

Trap. The formulas $s = r\theta$ and $A = (1/2)r^2\theta$ are only true with θ in radians. Plugging in 120° directly gives a wrong number. Convert to $2\pi/3$ first, or you are multiplying by the wrong unit entirely.

3.3 The Unit Circle

The **unit circle** is the circle of radius 1 centered at the origin. For an angle θ measured from the positive x-axis, the point on the circle is $(\cos \theta, \sin \theta)$. That makes cosine the x-coordinate and sine the y-coordinate, and it turns the five first-quadrant angles below into the exact values every later topic assumes you know.

θ (RADIANs)	$\sin \theta$	$\cos \theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Every row satisfies $\sin^2 \theta + \cos^2 \theta = 1$, which is a good way to check your memory: $(1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 1$ for $\pi/6$, and $(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2 = 1/2 + 1/2 = 1$ for $\pi/4$. A memory aid: the sine values across the row read $\sqrt{0}/2, \sqrt{1}/2, \sqrt{2}/2, \sqrt{3}/2, \sqrt{4}/2$, and the cosine values read the same list in reverse.

Angles outside the first quadrant come from **reference angles** plus quadrant signs. The sine keeps its sign pattern $++--$ across quadrants I through IV, and cosine keeps $+-+ -$. So $\cos(11\pi/6) = \cos(330^\circ)$: the reference angle is $\pi/6$ and the fourth quadrant makes cosine positive, giving $\sqrt{3}/2$.

Trap. The most swapped pair in the table is $\pi/6$ and $\pi/3$. Sine of $\pi/6$ is $1/2$ while sine of $\pi/3$ is $\sqrt{3}/2$. If you always mix them up, anchor on the 30-60-90 triangle: the side opposite 30° is the short side, $1/2$.

3.4 Sinusoidal Graphs: Amplitude, Period, Phase Shift, Midline

A **sinusoidal function** is any function that can be written $y = A \sin(B(x - C)) + D$, and the same form works with cosine. Each letter controls one visible feature. The **amplitude** is $|A|$, the height of the wave above and below the midline. The **period** is $2\pi/|B|$, the horizontal length of one full cycle. The **phase shift** is C , how far the graph slides horizontally, right when C is positive. The **midline** is the horizontal line $y = D$.

Work one example completely. For $y = 3 \sin(2(x - \pi/4)) + 1$: $A = 3$, so the amplitude is 3. $B = 2$, so the period is $2\pi/2 = \pi$, meaning the wave completes a full cycle every π units instead of every 2π . $C = \pi/4$, so the whole graph shifts right by $\pi/4$. $D = 1$, so the midline is $y = 1$ and the wave oscillates between $y = -2$ and $y = 4$.

Trap. The phase shift follows the sign inside the parentheses, not your instinct. The form is $(x - C)$, so $y = \sin(x - \pi/4)$ shifts *right* by $\pi/4$. Set the inside equal to zero: $x - \pi/4 = 0$ gives $x = \pi/4$, which is where the cycle now starts.

Trap. The period is $2\pi/B$, not B . For $y = \sin(2x)$, the period is $2\pi/2 = \pi$, a faster wave, not a slower one. Larger B squeezes the wave. A question that gives $B = 2$ and asks for the period is checking that you divided, not multiplied.

3.5 The Tangent Function

The **tangent function** is defined by $\tan \theta = \sin \theta / \cos \theta$, which means it is undefined wherever $\cos \theta = 0$. Its period is π , half the period of sine and cosine, because shifting by π flips the signs of both sine and cosine and the two negatives cancel in the ratio. The graph has zeros at every multiple of π and vertical asymptotes at $x = \pi/2 + n\pi$, and it increases across each interval between consecutive asymptotes.

3.6 Inverse Trigonometric Functions

Sine, cosine, and tangent repeat forever, so they have no true inverse until their domains are restricted to one monotonic piece. Each **inverse trigonometric function** undoes its partner only within that restricted range, and every calculator answer for an inverse trig function lands inside the range shown here. There are no exceptions to memorize beyond this table.

FUNCTION	DOMAIN	RANGE
$\arcsin x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	All real numbers	$(-\pi/2, \pi/2)$

Evaluate three examples, verifying each one. For $\arcsin(-\sqrt{3}/2)$: we need the angle in $[-\pi/2, \pi/2]$ whose sine is $-\sqrt{3}/2$, which is $-\pi/3$. Verify: $\sin(-\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$. For $\arccos(-1/2)$: we need the angle in $[0, \pi]$ whose cosine is $-1/2$, which is $2\pi/3$. Verify: $\cos(2\pi/3) = -\cos(\pi/3) = -1/2$. For $\arctan(1)$: we need the angle in $(-\pi/2, \pi/2)$ whose tangent is 1, which is $\pi/4$. Verify: $\tan(\pi/4) = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$.

Trap. An inverse trig answer is always checked against the range, not just the ratio. An angle like $5\pi/6$ can never be the value of $\arcsin$ or $\arctan$, no matter what the sine or tangent equals there. If your candidate is outside the range in the table, it is wrong.

3.7 Trigonometric Identities

Identities are equations that hold for every angle in their domains, and the CED tests them as equivalent representations of the same expression. The table below gives the full set for this course, each checked numerically at $\theta = \pi/6$ so you can see that both sides really do agree.

IDENTITY	STATEMENT	CHECK AT $\theta = \pi/6$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$(1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 1$
Reciprocal	$\csc\theta = 1/\sin\theta$, $\sec\theta = 1/\cos\theta$, $\cot\theta = 1/\tan\theta$	$\csc(\pi/6) = 1/(1/2) = 2$
Quotient	$\tan\theta = \sin\theta/\cos\theta$	$(1/2)/(\sqrt{3}/2) = 1/\sqrt{3} = \sqrt{3}/3$
Cofunction	$\sin(\pi/2 - \theta) = \cos\theta$	$\sin(\pi/2 - \pi/3) = \sin(\pi/6) = 1/2 = \cos(\pi/3)$
Even-odd	$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$	$\sin(-\pi/6) = -1/2 = -\sin(\pi/6)$
Periodicity	$\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + \pi) = \tan\theta$	$\sin(\pi/6 + 2\pi) = \sin(\pi/6) = 1/2$

Two extra Pythagorean forms come from dividing $\sin^2\theta + \cos^2\theta = 1$ through by $\cos^2\theta$ or $\sin^2\theta$: $\tan^2\theta + 1 = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$. They appear when a problem asks you to rewrite everything in terms of a single trig function.

Trap. The notation $\sin^2\theta$ means $(\sin\theta)^2$, and it is not $\sin(2\theta)$ and not $2\sin\theta$. Related: $\sin(A + B)$ is not $\sin A + \sin B$. The sine of a sum has its own structure, and distributing sine across a sum is one of the most penalized algebra errors in this unit.

3.8 Polar Coordinates

Polar coordinates locate a point by (r, θ) instead of (x, y) . The number r is the directed distance from the pole (the origin), and θ is the angle from the polar axis (the positive x -axis). The conversion formulas come straight from right-triangle trigonometry: $x = r \cos \theta$, $y = r \sin \theta$, and in the other direction $r^2 = x^2 + y^2$ with $\tan \theta = y/x$.

Convert $(r, \theta) = (4, \pi/3)$ to rectangular form. Then $x = 4 \cos(\pi/3) = 4 \times (1/2) = 2$, and $y = 4 \sin(\pi/3) = 4 \times (\sqrt{3}/2) = 2\sqrt{3}$. Verify with $r^2 = x^2 + y^2$: $2^2 + (2\sqrt{3})^2 = 4 + 12 = 16 = 4^2$. The conversion is consistent.

Convert $(x, y) = (3, -3)$ to polar form with $r \geq 0$. Then $r^2 = 9 + 9 = 18$, so $r = \sqrt{18} = 3\sqrt{2}$, and $\tan \theta = -3/3 = -1$. The point is in quadrant IV, so $\theta = -\pi/4$. Verify by converting back: $3\sqrt{2} \times \cos(-\pi/4) = 3\sqrt{2} \times \sqrt{2}/2 = 3$, and $3\sqrt{2} \times \sin(-\pi/4) = -3$. Both coordinates return exactly.

A **negative r** means walking the distance in the opposite direction of θ . The point $(-4, \pi/3)$ sits at distance 4 along the angle $\pi/3 + \pi = 4\pi/3$, which is the rectangular point $(-2, -2\sqrt{3})$. Check: $(-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16$.

Trap. The formula $\tan \theta = y/x$ does not finish the job by itself. Both $(3, -3)$ and $(-3, 3)$ give $\tan \theta = -1$, but they sit in different quadrants. Always place the point using the signs of x and y before choosing the angle.

3.9 Polar Graphs

Polar equations $r = f(\theta)$ trace curves as θ sweeps around, and the CED focuses on three families. **Circles:** $r = a$ is a circle of radius $|a|$ centered at the pole, while $r = 2a \cos \theta$ is a circle through the pole with diameter $2a$ along the polar axis. **Limaçons** have the form $r = a \pm b \cos \theta$ or $r = a \pm b \sin \theta$: a cardioid when $a = b$, a dimpled loop when $1 < a/b < 2$, an inner loop when $a/b < 1$, and a convex shape with no dimple when $a/b \geq 2$. For example, $r = 2 + 3 \cos \theta$ has $a/b = 2/3$, which is less than 1, so it has an inner loop.

Rose curves have the form $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$, and the **petal rule** counts the petals: if n is odd there are n petals, and if n is even there are $2n$ petals. Each petal extends $|a|$ from the pole. Verify the rule on two examples: $r = 5 \cos(3\theta)$ has $n = 3$, which is odd, so 3 petals of length 5. And $r = 4 \sin(2\theta)$ has $n = 2$, which is even, so $2 \times 2 = 4$ petals of length 4.

Trap. A negative r in a polar graph does not break the curve; it plots the point in the opposite direction. When tracing $r = 5 \cos(3\theta)$ by hand, the intervals where $\cos(3\theta)$ is negative still draw petals, just on the far side of the pole. Skipping negative r values leaves petals missing.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Period vs frequency	The period $2\pi/ B $ is the length of one cycle. The frequency $ B /2\pi$ is how many cycles fit in one unit of x . They are reciprocals.
Phase shift sign	The form is $(x - C)$, so a minus sign shifts right. Solve inside $= 0$ to find where the cycle now starts.
Degree vs radian mode	Every formula in this unit assumes radians. Confirm the calculator mode before computing, and convert degree inputs first.
Negative r in polar	Negative r means the opposite direction from θ , not a point that fails to exist. (r, θ) with $r < 0$ plots like $(r , \theta + \pi)$.
$\arcsin$ range vs $\arccos$ range	$\arcsin$ returns $[-\pi/2, \pi/2]$, which reaches into negative angles. $\arccos$ returns $[0, \pi]$, which never does. Match the answer to the right interval.
$\tan$ period vs $\sin/\cos$ period	Tangent repeats every π , while sine and cosine repeat every 2π . Asymptotes sit at $x = \pi/2 + n\pi$ for tangent.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. What is 225° in radians?

1. $4\pi/5$
2. $7\pi/4$
3. $3\pi/4$
4. $5\pi/4$

2. A sector of a circle has radius 6 and central angle $2\pi/3$. What is the length of the arc of the sector?

1. 4π
2. 2π
3. 8π
4. 12π

3. What is $\cos(11\pi/6)$?

1. $1/2$
2. $-\sqrt{3}/2$
3. $\sqrt{3}/2$
4. $-1/2$

4. What is the period of $y = 3 \sin(2(x - \pi/4)) + 1$?

1. 2π
2. π
3. $\pi/2$
4. 3π

5. What is $\arcsin(-\sqrt{3}/2)$?

1. $-\pi/3$
2. $-\pi/6$
3. $\pi/3$
4. $5\pi/6$

6. The point $(2, 2\sqrt{3})$ in rectangular coordinates corresponds to which polar point (r, θ) with $r \geq 0$?

1. $(2, \pi/3)$
2. $(4, 2\pi/3)$
3. $(4, \pi/3)$
4. $(4, \pi/6)$

7. The polar equation $r = 5 \cos(3\theta)$ graphs as a rose curve. How many petals does it have?

1. 2
2. 5
3. 6
4. 3

8. Which of the following is equivalent to $\tan \theta$ for all θ in its domain?

1. $\cos \theta / \sin \theta$
2. $\sin \theta / \cos \theta$
3. $1 / \sin \theta$
4. $\sin \theta \cdot \cos \theta$

Answer Key

- 1. D.** Multiply by $\pi/180$: $225 \times \pi/180 = 225\pi/180 = 5\pi/4$. A flips the conversion into $4\pi/5$, which is 144° . B is $7\pi/4$, which is 315° , a different quadrant entirely. C is $3\pi/4$, which is 135° .
- 2. A.** Arc length is $s = r\theta = 6 \times 2\pi/3 = 12\pi/3 = 4\pi$. B uses $\theta = \pi/3$, half the given angle. C doubles the correct answer, a sign of multiplying by r twice. D uses $\theta = 2\pi$, the full circle, instead of the given $2\pi/3$.
- 3. C.** The angle $11\pi/6$ is 330° , in quadrant IV with reference angle $\pi/6$, and cosine is positive in quadrant IV, so $\cos(11\pi/6) = \cos(\pi/6) = \sqrt{3}/2$. A confuses $\cos(\pi/6)$ with $\sin(\pi/6)$, which is $1/2$. B gets the magnitude right but the sign wrong; sine is negative in quadrant IV, cosine is not. D is $\sin(11\pi/6)$, the wrong coordinate.
- 4. B.** The period is $2\pi/|B| = 2\pi/2 = \pi$. A ignores the B value and reports the untransformed period of sine. C divides by B a second time. D folds the amplitude 3 into the period, but amplitude never affects horizontal length.
- 5. A.** $\arcsin$ returns angles in $[-\pi/2, \pi/2]$, and $\sin(-\pi/3) = -\sqrt{3}/2$, so $\arcsin(-\sqrt{3}/2) = -\pi/3$. B has the wrong magnitude: $\sin(-\pi/6) = -1/2$. C has the wrong sign: $\sin(\pi/3) = \sqrt{3}/2$, positive. D is outside the $\arcsin$ range entirely, and $\sin(5\pi/6) = 1/2$ anyway.
- 6. C.** $r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$, and $\tan \theta = 2\sqrt{3}/2 = \sqrt{3}$ with both coordinates positive, so $\theta = \pi/3$. A keeps $r = 2$, confusing the radius with the x-coordinate. B uses $2\pi/3$, the quadrant II supplement, but the point is in quadrant I. D uses $\pi/6$, the angle for the swapped point $(2\sqrt{3}, 2)$.
- 7. D.** For $r = a \cos(n\theta)$ with n odd, the rose has n petals, and here $n = 3$. A has no basis in the rule and may come from halving. B confuses the petal length $|a| = 5$ with the petal count. C applies the even- n rule ($2n$) to an odd n .
- 8. B.** The quotient identity gives $\tan \theta = \sin \theta / \cos \theta$. A is $\cot \theta$, the reciprocal of the correct answer. C is $\csc \theta$, the reciprocal of sine rather than the ratio of sine to cosine. D multiplies the two functions instead of dividing them.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. In Rycal, the Trigonometric and Polar Functions deck has flashcards for these ideas, and its practice questions target the same traps named here.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Trigonometric and Polar Functions deck and let spaced review bring them back over the next few days.

- Define a radian and state the conversion factors between degrees and radians.
- Convert 225° to radians and $11\pi/6$ radians to degrees from scratch, checking each result.
- State the arc length and sector area formulas and use them with $r = 6$ and $\theta = 2\pi/3$.
- Write the unit-circle table for 0 , $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$, and explain how symmetry extends it to the other quadrants.
- For $y = A \sin(B(x - C)) + D$, name the amplitude, period, phase shift, and midline in terms of A , B , C , and D .
- Describe the graph of $y = \tan x$: its period, its zeros, and where its asymptotes are.
- State the domain and range of $\arcsin$, $\arccos$, and $\arctan$.
- Evaluate $\arcsin(-\sqrt{3}/2)$, $\arccos(-1/2)$, and $\arctan(1)$, verifying each answer.
- State the Pythagorean, reciprocal, quotient, cofunction, even-odd, and periodicity identities.
- Convert between (r, θ) and (x, y) in both directions, and explain what a negative r means.
- Classify polar graphs as circles, limaçons, or rose curves, and state the petal rule.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Trigonometric and Polar Functions deck under AP Precalculus. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Radian, Degree-radian conversion, Arc length, Sector area, Unit circle, Reference angle, Amplitude, Period, Phase shift, Midline, Sinusoidal function, Tangent function, Inverse trigonometric function, $\arcsin$, $\arccos$, $\arctan$, Pythagorean identity, Reciprocal identity, Quotient identity, Cofunction identity, Even-odd identity, Periodicity, Polar coordinates, Pole, Polar axis, Limaçon, Cardioid, Rose curve, Petal rule.

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