

Unit 1: Polynomial and Rational Functions

Unit 1 is the foundation of AP Precalculus and the heaviest unit on the exam. It covers average rates of change, polynomial functions and their end behavior, zeros and multiplicity, polynomial division, and rational functions with their asymptotes.

AP Precalculus

Polynomial and Rational Functions

About 14 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. Average rate of change motivates the polynomial work, end behavior and zeros describe the shape of a polynomial, division and the Factor Theorem connect zeros to factors, and everything about polynomials carries over to rational functions, where the denominator adds holes and asymptotes. Exam questions often give a factored form or an expression and ask you to read the graph from it.

After the first read, use the trap boxes and the Confusions table to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Polynomial and Rational Functions is 30-40% of the AP Precalculus exam, the largest share of any unit. It also carries more weight than that number suggests, because these functions underpin everything later in the course. Exponential, logarithmic, and trigonometric models all lean on the polynomial and rational behavior you learn here. That makes this the unit to get solid first.

Average Rate of Change

The **average rate of change** of a function f over the interval $[a, b]$ is $[f(b) - f(a)] / (b - a)$. It is the change in output divided by the change in input. On a graph it is the slope of the **secant line** through the two points $(a, f(a))$ and $(b, f(b))$. It tells you how fast the output changes per unit of input, on average, across the whole interval.

A worked example, computed fully. Let $f(x) = x^2$, and take the interval $[1, 3]$. First find the outputs. $f(1) = 1^2 = 1$, and $f(3) = 3^2 = 9$. Now form the fraction. The change in output is $9 - 1 = 8$. The change in input is $3 - 1 = 2$. So the average rate of change is $8/2 = 4$. The secant line through $(1, 1)$ and $(3, 9)$ has slope 4.

Trap. An average rate belongs to an interval, not a point. The value 4 above says nothing about the rate at $x = 2$ by itself. Also, a question that says "over the interval $[1, 3]$ " wants the full fraction, not just the numerator 8 and not just the denominator 2.

Polynomial Functions

A **polynomial function** has the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where the exponents are whole numbers and a_n is not zero. The **degree** n is the largest exponent. The **leading coefficient** a_n is the coefficient of that term. For $f(x) = -2x^3 + x^2 + 5$, the degree is 3 and the leading coefficient is -2 .

The degree and the leading coefficient control the end behavior. The zeros and factors control where the graph meets the x -axis. Keep these two jobs separate, because exam questions test whether you mix them up.

End Behavior and the Leading-Coefficient Test

End behavior is what happens to $f(x)$ as x goes to positive or negative infinity. For a polynomial, only the leading term $a_n x^n$ matters far out, because it grows faster than every other term combined. The degree tells you whether the two ends match or point in opposite directions. The sign of the leading coefficient tells you which way they go.

CASE	END BEHAVIOR	EXAMPLE
Even degree, positive leading coefficient	Both ends go up. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$, and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$.	$f(x) = x^2$
Even degree, negative leading coefficient	Both ends go down. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$, and as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.	$f(x) = -x^2$
Odd degree, positive leading coefficient	Ends go opposite ways, up on the right. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$, and as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.	$f(x) = x^3$
Odd degree, negative leading coefficient	Ends go opposite ways, down on the right. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$, and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$.	$f(x) = -x^3$

Example. For $f(x) = -2x^3 + x^2 + 5$, the degree is 3 (odd) and the leading coefficient is -2 (negative). This is the fourth row of the table. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$, and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. The $+5$ and the x^2 term do not change this, because far out the $-2x^3$ term dominates everything else.

Trap. End behavior is set by the leading term alone. The zeros of the function and the constant term do not decide the ends. A cubic with a positive leading coefficient always exits up on the right, no matter how many zeros it has or where they sit.

Zeros, Multiplicity, and Turning Points

A **zero** of f is an input c where $f(c) = 0$. Every real zero is an x-intercept of the graph. The **multiplicity** of a zero is the number of times its factor appears in the factored form. In $f(x) = (x - 2)^2(x + 1)$, the zero $x = 2$ has multiplicity 2, and the zero $x = -1$ has multiplicity 1.

Multiplicity decides how the graph meets the x-axis. A zero of **odd** multiplicity makes the graph **cross** the axis, changing sign from one side to the other. A zero of **even** multiplicity makes the graph **touch** the axis and turn back, without changing sign. In the example, the graph crosses the axis at $x = -1$ and touches the axis and turns around at $x = 2$.

A **turning point** is a spot where the graph switches from increasing to decreasing, or from decreasing to increasing. A polynomial of degree n has at most $n - 1$ turning points. A cubic has at most 2, a quartic at most 3, and so on. The words "at most" matter, because some polynomials have fewer than the maximum.

Trap. A zero of multiplicity 2 is still a zero, and the graph still touches the axis there. A common mistake is to read "touches" as "not really a zero." It is a zero. It simply does not cross.

Factor Theorem and Remainder Theorem

The **Factor Theorem** says that $(x - c)$ is a factor of a polynomial f exactly when $f(c) = 0$. It connects zeros and factors in both directions. Finding a zero hands you a factor, and finding a factor hands you a zero. If you know $x = -1$ is a zero, then $(x + 1)$ is a factor, and you can divide it out to find the rest.

The **Remainder Theorem** says that when f is divided by $(x - c)$, the remainder equals $f(c)$. This gives a quick check on any division you do. On the next page, dividing $x^3 - 3x^2 + 4$ by $(x - 2)$ leaves remainder 0. The theorem agrees, because $f(2) = 2^3 - 3(2^2) + 4 = 8 - 12 + 4 = 0$.

For polynomials with real coefficients, complex zeros come in **conjugate pairs**. If $2 + 3i$ is a zero, then $2 - 3i$ must also be a zero. Their factors multiply to a real quadratic. $(x - (2 + 3i))(x - (2 - 3i)) = x^2 - 4x + 13$, which has all real coefficients. A degree- n polynomial has exactly n zeros when you count multiplicity and include the complex ones, so an odd-degree polynomial with real coefficients always has at least one real zero.

Trap. The conjugate-pair rule needs real coefficients. If the coefficients are not all real, a complex zero can appear alone. On the AP exam the polynomials have real coefficients unless the question says otherwise, so look for the conjugate.

Polynomial Division

Synthetic division is the short form for dividing by $(x - c)$. Write the coefficients across, bring down the first, multiply by c , add to the next coefficient, and repeat down the row. Here is the division of $x^3 - 3x^2 + 4$ by $(x - 2)$, done one step at a time. The divisor $(x - 2)$ gives $c = 2$, and the coefficients are 1, -3, 0, 4. The 0 stands in for the missing x term, and leaving it out is a classic error.

Step by step. Bring down the 1. Multiply $2 \times 1 = 2$ and write it under -3. Add: $-3 + 2 = -1$. Multiply $2 \times (-1) = -2$ and write it under 0. Add: $0 + (-2) = -2$. Multiply $2 \times (-2) = -4$ and write it under 4. Add: $4 + (-4) = 0$. The finished bottom row is 1, -1, -2, 0.

Read the result as a quotient and a remainder. The bottom row except the last number gives the coefficients of the quotient, one degree lower: $x^2 - x - 2$. The last number is the remainder, 0. So $x^3 - 3x^2 + 4 = (x - 2)(x^2 - x - 2)$, and the quotient factors further as $(x - 2)(x + 1)$. The full factorization is $(x - 2)^2(x + 1)$, which shows the zero $x = 2$ with multiplicity 2 and the zero $x = -1$ with multiplicity 1.

Check the division by multiplying back. $(x - 2)(x^2 - x - 2) = x^3 - x^2 - 2x - 2x^2 + 2x + 4 = x^3 - 3x^2 + 4$. The product matches the original polynomial. The Remainder Theorem agrees too, since $f(2) = 8 - 12 + 4 = 0$.

Trap. The sign of c is the most common arithmetic error in synthetic division. The divisor $(x - 2)$ gives $c = 2$, but the divisor $(x + 2)$ gives $c = -2$. Write c down before you start multiplying. Then verify at the end by multiplying the quotient by the divisor and adding the remainder.

Rational Functions: Domain, Holes, and Vertical Asymptotes

A **rational function** is a ratio of two polynomials, $f(x) = p(x) / q(x)$, where q is not the zero polynomial. The **domain** is all real numbers except where the denominator equals zero. Start every rational-function problem by finding those excluded values.

Factor both polynomials first. A factor that appears in both the numerator and the denominator cancels, and each canceled factor at $x = c$ produces a **hole** (a removable discontinuity) at that x -value. A factor that remains in the denominator gives a **vertical asymptote** at $x = c$, where the function grows without bound.

Example. Let $f(x) = (x^2 - 9) / (x^2 - 2x - 15)$. Factoring gives $(x - 3)(x + 3)$ over $(x - 5)(x + 3)$. The factor $(x + 3)$ cancels, so there is a hole at $x = -3$. The simplified form is $(x - 3) / (x - 5)$, and the hole sits at $(-3, 3/4)$, because $(-3 - 3) / (-3 - 5) = (-6) / (-8) = 3/4$. The remaining denominator factor $(x - 5)$ gives a vertical asymptote at $x = 5$.

Behavior near the asymptote needs a **sign analysis**. Just left of $x = 5$, say $x = 4.9$, the simplified form gives $(4.9 - 3) / (4.9 - 5) = 1.9 / (-0.1) = -19$, so $f(x) \rightarrow -\infty$ from the left. Just right of $x = 5$, at $x = 5.1$, it gives $2.1 / 0.1 = 21$, so $f(x) \rightarrow +\infty$ from the right. Test one value on each side instead of guessing the direction.

Trap. A canceled factor means a hole, not a vertical asymptote. The factor $(x + 3)$ above is gone from the simplified expression, so it cannot make the function blow up. If you list $x = -3$ as an asymptote, you treated a hole as an asymptote.

Horizontal Asymptotes

A **horizontal asymptote** describes what $f(x)$ approaches as $x \rightarrow \infty$ or $x \rightarrow -\infty$. For a rational function, compare the degree of the numerator with the degree of the denominator. There are three cases.

CASE	HORIZONTAL ASYMPTOTE	EXAMPLE
Degree of numerator is less than degree of denominator	$y = 0$. The denominator outgrows the numerator.	$(2x + 1) / (x^2 - 4)$: $y = 0$
Degrees are equal	$y = (\text{leading coefficient of numerator}) / (\text{leading coefficient of denominator})$.	$(3x^2 + 1) / (2x^2 - 5)$: $y = 3/2$
Degree of numerator is greater than degree of denominator	No horizontal asymptote. The function grows without bound, or it follows a slant asymptote.	$(x^2 + 1) / (x - 1)$: no horizontal asymptote

Check the equal-degree case by dividing top and bottom by the highest power. For $(3x^2 + 1) / (2x^2 - 5)$, dividing by x^2 gives $(3 + 1/x^2) / (2 - 5/x^2)$. As $x \rightarrow \infty$, the $1/x^2$ terms go to 0, leaving $3/2$. So the graph settles toward $y = 3/2$ at both ends.

Trap. Pick the degree case before you compute anything. For $(3x^2 + 1) / (2x^2 - 5)$ the degrees are equal, so the answer is the ratio $3/2$, not the numerator's coefficient alone and not $y = 0$. And a graph may cross its horizontal asymptote. The asymptote only says where the function settles far out, so a question about a finite crossing needs the equation $(3x^2 + 1) / (2x^2 - 5) = 3/2$, not the asymptote rule.

Slant Asymptotes and Behavior Near Asymptotes

When the numerator degree is exactly one more than the denominator degree, the graph approaches a slanted line instead of a horizontal one. Divide the polynomials, and the quotient is the **slant asymptote**, because the remainder term shrinks to zero as x grows.

Example. For $f(x) = (x^2 + 1) / (x - 1)$, the numerator degree 2 is exactly one more than the denominator degree 1. Division gives quotient $x + 1$ with remainder 2, so $f(x) = x + 1 + 2 / (x - 1)$. As $x \rightarrow \pm\infty$, the term $2 / (x - 1)$ goes to 0, and the graph settles toward the line $y = x + 1$. That line is the slant asymptote.

This function also has a vertical asymptote at $x = 1$. A sign check shows $f(x) \rightarrow -\infty$ from the left, since at $x = 0.9$ the value is $(0.81 + 1) / (0.9 - 1) = 1.81 / (-0.1) = -18.1$, and $f(x) \rightarrow +\infty$ from the right, since at $x = 1.1$ the value is $2.21 / 0.1 = 22.1$.

The remainder term also tells you which side of the slant line the graph sits on. For large positive x , $2 / (x - 1)$ is positive, so the graph sits just above $y = x + 1$. For large negative x , it is negative, so the graph sits just below. You do not need this for most questions, but it settles arguments about the sketch.

Trap. A slant asymptote appears only when the numerator degree is exactly one more than the denominator degree. If it is two or more higher, there is no linear asymptote at all. A graph has a horizontal asymptote or a slant asymptote, never both.

Confusions That Cost Points

These are the pairs that exam questions use to separate careful readers from fast ones. For each pair, make sure you can state the difference out loud.

PAIR	HOW TO KEEP THEM STRAIGHT
Zeros vs x-intercepts with multiplicity	Every real zero is an x-intercept, but multiplicity changes the meeting. Odd multiplicity crosses the axis. Even multiplicity touches and turns back. The zero is real in both cases.
Holes vs vertical asymptotes	A canceled factor gives a hole at a finite point. A denominator factor that survives gives a vertical asymptote. Factor fully, cancel, then classify what is left.
End behavior vs zeros	The leading term decides the ends. The factored form decides the zeros. A degree-5 polynomial with five zeros still follows its leading term far out.
Horizontal vs slant asymptote	Equal degrees give $y =$ the ratio of the leading coefficients. A numerator exactly one degree higher gives the quotient line as a slant asymptote. A graph has one or the other, never both.
Average rate of change vs a function value	The rate over $[a, b]$ is $[f(b) - f(a)] / (b - a)$, one number for the whole interval. It is not f at either endpoint, and it is not the numerator alone.
Factor Theorem vs Remainder Theorem	The Factor Theorem turns a zero into a factor: $f(c) = 0$ means $(x - c)$ is a factor. The Remainder Theorem turns a division into an evaluation: the remainder of f divided by $(x - c)$ is $f(c)$.

Read this table once more before you start the practice questions. Three of the eight questions below are built directly on these pairs.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Let $f(x) = x^2$. What is the average rate of change of f over the interval $[1, 3]$?
 1. 8
 2. 4
 3. 2
 4. 5

2. Let $f(x) = -2x^3 + x^2 + 5$. As $x \rightarrow -\infty$, $f(x) \rightarrow$
 1. ∞
 2. $-\infty$
 3. 0
 4. 5

3. The graph of a polynomial function has a zero of multiplicity 2 at $x = 3$. Which statement describes the graph at $x = 3$?
 1. The graph crosses the x -axis there, changing sign.
 2. The graph has a hole at $(3, 0)$.
 3. The graph has a vertical asymptote at $x = 3$.
 4. The graph touches the x -axis there and turns back without changing sign.

4. When $2x^3 - 5x^2 + x + 2$ is divided by $(x - 2)$, the remainder is
 1. -1
 2. 2
 3. 0
 4. 1

5. Let $f(x) = (x^2 - 1) / (x^2 - 3x + 2)$. Which statement is true?

1. f has a vertical asymptote at $x = 1$ and a hole at $x = 2$.
2. f has a hole at $x = 1$ and a vertical asymptote at $x = 2$.
3. f has vertical asymptotes at $x = 1$ and $x = 2$.
4. The domain of f is all real numbers.

6. Let $f(x) = (4x^2 - x) / (2x^2 + 7)$. Which line is a horizontal asymptote of f ?

1. $y = 2$
2. $y = 0$
3. $y = 4$
4. f has no horizontal asymptote

7. Let $f(x) = (x^2 + 1) / (x - 1)$. Which line is the slant asymptote of f ?

1. $y = 0$
2. $y = 1$
3. $y = x + 1$
4. f has no slant asymptote

8. A polynomial with real coefficients has $2 + 3i$ as a zero. Which of the following must also be a zero?

1. $3 + 2i$
2. $-2 - 3i$
3. $2i$
4. $2 - 3i$

Answer Key

- 1. B.** The average rate is $[f(3) - f(1)] / (3 - 1) = (9 - 1) / 2 = 8/2 = 4$. A is the change in output with the division step skipped. C is the change in input alone, which is not a rate. D averages the outputs, $(1 + 9) / 2$, instead of differencing them.
- 2. A.** The degree 3 is odd and the leading coefficient -2 is negative, so the ends point in opposite directions with the left end up: as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. B is the behavior as $x \rightarrow \infty$, the wrong end. C applies the rational-function rule for a smaller numerator degree, which does not apply to polynomials. D confuses the y-intercept $f(0) = 5$ with end behavior.
- 3. D.** Even multiplicity means the graph touches the x-axis and turns back without changing sign. A describes odd multiplicity. B confuses a zero with a hole; the function value is 0 there, not undefined. C confuses a zero with a vertical asymptote.
- 4. C.** Synthetic division with $c = 2$ on the coefficients 2, -5 , 1, 2 gives the bottom row 2, -1 , -1 , 0, so the remainder is 0. The Remainder Theorem agrees: $f(2) = 16 - 20 + 2 + 2 = 0$. A is the last quotient coefficient, from stopping one column early. B is the first quotient coefficient. D comes from an arithmetic or sign slip in the multiply-and-add steps.
- 5. B.** Factoring gives $(x - 1)(x + 1)$ over $(x - 1)(x - 2)$. The canceled factor $(x - 1)$ produces a hole at $x = 1$, and the remaining denominator factor gives a vertical asymptote at $x = 2$. A swaps the two. C treats the canceled factor as an asymptote. D ignores that the denominator is zero at $x = 1$ and $x = 2$.
- 6. A.** The degrees are both 2, so the horizontal asymptote is the ratio of the leading coefficients, $4/2 = 2$. B would be correct only if the denominator degree were larger. C uses only the numerator's leading coefficient. D applies the rule for a larger numerator degree, but the degrees are equal here.
- 7. C.** The numerator degree 2 is exactly one more than the denominator degree 1. Division gives $(x^2 + 1) = (x - 1)(x + 1) + 2$, so $f(x) = x + 1 + 2 / (x - 1)$, and the remainder term vanishes as x grows, leaving $y = x + 1$. A applies the horizontal rule for a smaller numerator degree. B mistakes the vertical asymptote $x = 1$ for the slant one. D would be right only if the numerator degree were at least two more than the denominator's.
- 8. D.** With real coefficients, complex zeros come in conjugate pairs, so $2 + 3i$ forces $2 - 3i$. A swaps the real and imaginary parts. B negates both parts. C drops the real part entirely.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Polynomial and Rational Functions deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Polynomial and Rational Functions deck and let spaced review bring them back over the next few days.

- State the average rate of change formula and explain what the secant line represents.
- Explain how degree and leading coefficient together determine end behavior, for even and odd degrees.
- State the four end-behavior cases from memory, with an example of each.
- Explain how multiplicity decides whether a graph crosses or touches the x-axis at a zero.
- State the bound on the number of turning points for a polynomial of degree n .
- State the Factor Theorem and use it to explain why $(x - 2)$ is a factor of $x^3 - 3x^2 + 4$.
- State the Remainder Theorem and use it to check a synthetic division.
- Explain why complex zeros of a real-coefficient polynomial come in conjugate pairs.
- Carry out a synthetic division from scratch, showing every multiply-and-add step.
- Find the hole and the vertical asymptote of $(x^2 - 9) / (x^2 - 2x - 15)$, including the hole's coordinates.
- State the three degree-comparison cases for horizontal asymptotes.
- Find the slant asymptote of $(x^2 + 1) / (x - 1)$ and explain why the remainder term vanishes.
- Describe how to do a sign analysis on each side of a vertical asymptote.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Polynomial and Rational Functions deck under AP Precalculus. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Average rate of change, Secant line, Polynomial function, Degree, Leading coefficient, End behavior, Leading-coefficient test, Zero, Multiplicity, Turning point, Factor Theorem, Remainder Theorem, Synthetic division, Conjugate pairs, Rational function, Domain, Hole (removable discontinuity), Vertical asymptote, Horizontal asymptote, Slant asymptote, Sign analysis.

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