

Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions

Unit 9 gives you new ways to describe curves. Parametric equations trace a curve through time, vector-valued functions package that motion for physics-style problems, and polar coordinates describe curves by angle and distance from the origin. The calculus you already know applies to all three, with a few new formulas.

AP Calculus BC

Parametric, Polar, and Vector Functions

About 14 minutes to read

How to use this guide

Read it in order the first time. Parametric equations come first because everything else reuses their ideas: vector-valued functions are parametric equations written as a single object, and polar curves are differentiated with the same chain-rule pattern. The polar area formula is the one genuinely new integral here. After the first read, use the trap boxes for the distinctions the exam tests most, then work the practice questions before checking the answer key.

What this unit is worth. Parametric, polar, and vector topics are about 11 to 15 percent of the AP Calculus BC exam, and they appear nowhere on the AB exam. Every question here is BC-only, so this unit is one of the clearest places your BC score comes from. The most-tested skills are the parametric second derivative, polar area, and particle motion with vectors.

9.1 Defining and Differentiating Parametric Equations

Parametric equations describe a curve with two functions of a third variable, usually t : $x = x(t)$, $y = y(t)$. As t varies, the point $(x(t), y(t))$ traces the curve. Think of t as time and the point as a particle moving along the path. The curve $x = t^2$, $y = t^3$ is traced as t runs through its domain, and the direction of motion matters: the same geometric curve traced in the opposite direction is a different parametrization.

Eliminating the parameter means rewriting the pair as one equation in x and y alone, often by solving one equation for t and substituting into the other. For $x = t^2$, $y = t^3$, solving gives $t = \sqrt{x}$ (for $t \geq 0$), so $y = x^{3/2}$. Eliminating the parameter is useful for recognizing the curve, but it can lose information: the parametrization also records direction and speed, which the bare equation $y = x^{3/2}$ does not.

The **derivative of a parametric function** gives the slope of the tangent line at each point: $dy/dx = (dy/dt)/(dx/dt)$, provided $dx/dt \neq 0$. This is the chain rule in disguise, since $dy/dx = (dy/dt) \cdot (dt/dx)$. For $x = t^2$, $y = t^3$: $dy/dt = 3t^2$ and $dx/dt = 2t$, so $dy/dx = 3t^2/(2t) = 3t/2$. At $t = 2$ the slope is 3.

Trap. The quotient $dy/dx = (dy/dt)/(dx/dt)$ is not the derivative of a fraction. It is the chain rule: how fast y changes with t divided by how fast x changes with t . If you need dy/dx at a point, you need both derivatives at the same t value, not at the same x or y .

9.2 Second Derivatives of Parametric Equations

The **second derivative of a parametric function** measures concavity, but it is not d^2y/dt^2 divided by anything simple. The correct formula is $d^2y/dx^2 = [d/dt(dy/dx)]/(dx/dt)$: differentiate dy/dx with respect to t first, then divide by dx/dt . The reason is the same chain rule as before, applied to the first derivative.

For $x = t^2$, $y = t^3$: $dy/dx = 3t/2$, so $d/dt(dy/dx) = 3/2$, and dividing by $dx/dt = 2t$ gives $d^2y/dx^2 = 3/(4t)$. At $t = 2$ the second derivative is $3/8$, positive, so the curve is concave up there. Notice the second derivative is a function of t , just like the first: parametric concavity is evaluated at a parameter value, not at an x value.

Trap. The most common error in this unit is computing d^2y/dt^2 and calling it the second derivative. That measures how the y -component accelerates in time, which is not concavity. Concavity is about how the slope dy/dx changes as x changes, so you must differentiate dy/dx with respect to t and then divide by dx/dt .

9.3 Finding Arc Lengths of Curves Given by Parametric Equations

The **arc length of a parametric curve** from $t = t_1$ to $t = t_2$ is $\int \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$. The idea is the same as the Cartesian arc length formula: chop the curve into tiny segments, approximate each with the distance formula, and integrate. The expression under the square root is the speed of the particle, so arc length is the integral of speed, which is the total distance traveled.

For the unit circle $x = \cos t$, $y = \sin t$ from $t = 0$ to $t = 2\pi$: $dx/dt = -\sin t$ and $dy/dt = \cos t$, so the integrand is $\sqrt{(\sin^2 t + \cos^2 t)} = 1$, and the arc length is $\int_0^{2\pi} 1 dt = 2\pi$, the circumference. On the exam this integral is almost always evaluated numerically, so the skill being tested is setting it up correctly, not computing it by hand.

Trap. Arc length uses the parameter interval, not x-values, as the limits of integration. If the curve is traced from $t = 0$ to $t = \pi$, those are your limits even if x runs from -1 to 1 . Also, if the particle retraces part of the curve, the integral counts that distance twice, which is correct for total distance but not for displacement.

9.4 Defining and Differentiating Vector-Valued Functions

A **vector-valued function** $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ packages a parametric curve as a single object: a pair of ordinary functions of t that together trace a curve in the plane. The notation is new but the content is the same as parametric equations. The advantage is that calculus operations apply to the whole vector at once.

The **derivative of a vector-valued function** is $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$: differentiate each component with the usual rules. For $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle$. Geometrically, $\mathbf{r}'(t)$ is the velocity vector, tangent to the curve and pointing in the direction of motion.

9.5 Integrating Vector-Valued Functions

The **integral of a vector-valued function** works component by component: integrate each function with the usual antiderivative rules. A definite integral $\int_a^b \mathbf{r}(t) \, dt$ gives a vector whose components are the ordinary definite integrals.

Position from a velocity vector is the main application. Given a particle's velocity $\mathbf{v}(t)$ and its starting position $\mathbf{r}(t_0)$, integrate the velocity and use the starting point to find the constant: $\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^t \mathbf{v}(s) \, ds$. This is an initial value problem in the plane. For $\mathbf{v}(t) = \langle 2t, 3t^2 \rangle$ with $\mathbf{r}(0) = \langle 1, 0 \rangle$: integrating gives $\langle t^2 + C_1, t^3 + C_2 \rangle$, and $\mathbf{r}(0) = \langle 1, 0 \rangle$ forces $C_1 = 1, C_2 = 0$, so $\mathbf{r}(t) = \langle t^2 + 1, t^3 \rangle$.

Trap. The constant of integration is a vector, not a number: each component gets its own constant. When the problem gives a starting position, use it to solve for both constants separately. Forgetting one component's constant is the same error as forgetting $+C$ in ordinary integration, done twice.

9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

For a particle moving in the plane, the **velocity vector** is $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$, the time derivative of position. It encodes both rate and direction. The **speed** is the nonnegative scalar $|\mathbf{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$, the magnitude of the velocity vector. Speed is a number; velocity is a vector. For $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ at $t = 2$: $\mathbf{v}(2) = \langle 4, 12 \rangle$ and speed $= \sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$.

Acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \langle x''(t), y''(t) \rangle$, the rate of change of velocity. A useful fact: if speed is increasing, the velocity and acceleration vectors point in roughly the same direction (their dot product is positive); if speed is decreasing, they point in roughly opposite directions.

Displacement versus distance traveled is the distinction the exam returns to again and again. Over a time interval, the displacement is the net change in position, found by integrating the velocity vector: $\int \mathbf{v}(t) \, dt$. The distance traveled is the total path length, found by integrating the speed: $\int |\mathbf{v}(t)| \, dt$. Displacement can be zero for a round trip; distance never is.

Trap. Integrating velocity gives displacement, not distance. If a particle moves right 5 units then left 3, the displacement is 2 but the distance traveled is 8. Whenever the velocity changes sign inside the interval, split the integral of speed at the turning points.

9.7 Defining Polar Coordinates and Differentiating in Polar Form

Polar coordinates locate a point by (r, θ) : r is the distance from the origin and θ is the angle from the positive x -axis. The conversion equations are $x = r \cos \theta$, $y = r \sin \theta$, and $r^2 = x^2 + y^2$. A polar curve $r = f(\theta)$ is traced as θ varies, and unlike Cartesian graphs, the same point can have many polar representations: (r, θ) and $(-r, \theta + \pi)$ name the same point.

The **derivative in polar form** for $r = f(\theta)$ is $dy/dx = (dy/d\theta)/(dx/d\theta)$, the same chain-rule pattern as parametric equations, because a polar curve is a parametric curve with $x = r(\theta)\cos \theta$ and $y = r(\theta)\sin \theta$. The tangent behavior follows from the two θ -derivatives: zeros of $dy/d\theta$ locate **horizontal tangents**, and points where $dx/d\theta = 0$ give **vertical tangents**. The sign of $dr/d\theta$ shows where r grows or shrinks as θ increases.

Take the cardioid $r = 1 + \cos \theta$. Then $y = (1 + \cos \theta) \sin \theta$ and $dy/d\theta = \cos \theta + \cos 2\theta$, which is zero at $\theta = \pi/3, \pi$, and $5\pi/3$. Meanwhile $x = (1 + \cos \theta) \cos \theta$ and $dx/d\theta = -\sin \theta(1 + 2\cos \theta)$, zero at $\theta = 0, 2\pi/3, \pi, 4\pi/3$, and 2π . At $\theta = \pi/3$, $dy/d\theta = 0$ while $dx/d\theta = -\sqrt{3} \neq 0$, so the tangent is genuinely horizontal. At $\theta = \pi$ both derivatives are zero: that is the cusp at the origin, where the curve comes to a sharp point and the tangent is undefined.

Trap. Zeros of $dy/d\theta$ give horizontal tangents only when $dx/d\theta$ is nonzero at the same point. If both are zero, you have a cusp or a self-intersection, not a horizontal tangent. Always check $dx/d\theta$ before concluding. Also, $dr/d\theta = 0$ does not locate horizontal tangents; it locates where r stops growing, which is a different question.

9.8 Finding the Area of a Polar Region

The **area of a polar region** swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is $(1/2)\int_{\alpha}^{\beta} r^2 d\theta$. The formula comes from adding up thin sectors: a sector of radius r and angle $d\theta$ has area $(1/2)r^2 d\theta$, and integrating sweeps the region. The $1/2$ is the most commonly forgotten factor on the exam.

For the cardioid $r = 1 + \cos \theta$ traced over 0 to 2π : area $= (1/2)\int_0^{2\pi} (1 + \cos \theta)^2 d\theta$. Expanding gives $(1/2)\int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$. The $\cos \theta$ term integrates to zero over a full period, and $\int_0^{2\pi} \cos^2 \theta d\theta = \pi$, so the area is $(1/2)(2\pi + 0 + \pi) = 3\pi/2$. The θ -limits must cover the curve exactly once: for this cardioid, 0 to 2π traces the full loop.

Trap. The θ -interval must trace the region exactly once. Some curves retrace themselves: $r = \cos \theta$ over 0 to 2π traces the same circle twice, so the correct interval for the area is 0 to π . If you integrate over the doubled interval, you double-count the area. Sketch or reason about the tracing before choosing limits.

9.9 Finding the Area of the Region Bounded by Two Polar Curves

The **area between two polar curves** is $A = (1/2)\int (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta$, taken over the θ -interval where one graph lies outside the other. The intersections of the curves set the limits of integration. This is the polar version of the familiar top-minus-bottom area formula, with the $1/2$ from the sector area.

The subtlety is that polar intersections are trickier than Cartesian ones. Two polar curves can intersect at points whose θ -values differ: $r = 1$ and $r = \cos \theta$ meet at the origin, but the origin is $(0, \text{any } \theta)$ on the first curve and $(0, \pi/2)$ on the second. Solve $f(\theta) = g(\theta)$ for candidate angles, then verify each candidate is actually a common point before using it as a limit.

Trap. Setting the two r -expressions equal finds some intersections but can miss ones at the origin, since the origin has no single θ . Always check whether either curve passes through the origin on your interval, and sketch the region to confirm which curve is outer before subtracting.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
dy/dx vs d^2y/dx^2 (parametric)	$dy/dx = (dy/dt)/(dx/dt)$. For the second, differentiate dy/dx with respect to t , then divide by dx/dt again. Never use d^2y/dt^2 alone.
$dy/d\theta = 0$ vs $dx/d\theta = 0$ (polar)	Zeros of $dy/d\theta$ give horizontal tangents; zeros of $dx/d\theta$ give vertical tangents. If both are zero at once, it is a cusp, not a tangent of either kind.
$dr/d\theta = 0$ vs $dy/d\theta = 0$	$dr/d\theta = 0$ is where r stops growing. Horizontal tangents need $dy/d\theta = 0$. Different derivatives, different answers.
Displacement vs distance traveled	Displacement is $\int v(t) dt$ (net change, a vector). Distance is $\int v(t) dt$ (total path, a scalar). Split at turning points for distance.
Speed vs velocity	Velocity $\langle x', y' \rangle$ is a vector with direction. Speed $ v $ is its magnitude, a nonnegative number.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. For the curve defined by $x = t^2$, $y = t^3$, what is dy/dx at $t = 2$?

- A. $3/4$
- B. 3
- C. 6
- D. 12

2. For the same curve $x = t^2$, $y = t^3$, what is d^2y/dx^2 at $t = 2$?

- A. $3/8$
- B. $3/2$
- C. $3t$
- D. 0

3. What is the area enclosed by the polar curve $r = 1 + \cos \theta$?

- A. π
- B. $3\pi/2$
- C. 2π
- D. 3π

4. For the cardioid $r = 1 + \cos \theta$, what is true at $\theta = \pi/3$?

- A. The tangent line is horizontal.
- B. The tangent line is vertical.
- C. The curve has a cusp.
- D. r attains its maximum value.

Answer Key

- 1. B.** $dy/dx = (dy/dt)/(dx/dt) = 3t^2/(2t) = 3t/2$. At $t = 2$ this is 3. A comes from substituting $t = 2$ into the denominator only, computing $3/(2 \cdot 2) = 3/4$ and forgetting the t in the numerator of $3t/2$. C forgets to divide by dx/dt and reports $dy/dt = 3t^2 = 12$ at $t = 2$, then halves it incorrectly. D reports dy/dt alone.
- 2. A.** First $dy/dx = 3t/2$. Then $d/dt(dy/dx) = 3/2$, and dividing by $dx/dt = 2t$ gives $d^2y/dx^2 = 3/(4t)$. At $t = 2$ this is $3/8$. B is dy/dx evaluated at $t = 1$, a wrong-parameter error. C differentiates y twice with respect to t ($6t$) and drops the chain-rule division entirely. D assumes the second derivative of a cubic-looking curve is zero, confusing parametric concavity with polynomial degree.
- 3. B.** Area $= (1/2) \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = (1/2)(2\pi + \pi) = 3\pi/2$. A forgets the $\cos^2 \theta$ contribution and integrates only the constant term. C forgets both the $1/2$ in the area formula and the square on r , computing $\int_0^{2\pi} (1 + \cos \theta) d\theta = 2\pi$. D drops the $1/2$ factor, the single most common polar area error.
- 4. A.** $dy/d\theta = \cos \theta + \cos 2\theta = 0$ at $\theta = \pi/3$, while $dx/d\theta = -\sin \theta(1 + 2\cos \theta) = -\sqrt{3} \neq 0$ there, so the tangent is horizontal. B would require $dx/d\theta = 0$ with $dy/d\theta \neq 0$, which happens at $\theta = 2\pi/3$, not $\pi/3$. C requires both derivatives zero, which happens at $\theta = \pi$ (the cusp at the origin). D is wrong because $r = 1 + \cos \theta$ is maximal at $\theta = 0$, where $r = 2$.

One-Page Recall Check

- Define parametric equations and explain what eliminating the parameter does and does not preserve.
- Write the formula for dy/dx for $x = x(t)$, $y = y(t)$, and explain why it is the chain rule.
- Write the formula for d^2y/dx^2 and explain why d^2y/dt^2 alone is wrong.
- Compute dy/dx and d^2y/dx^2 at $t = 1$ for $x = t^2$, $y = t^3$.
- Write the parametric arc length formula and explain what the integrand represents.
- Define a vector-valued function and give its derivative.
- Explain how to recover position from a velocity vector and an initial position.
- Distinguish velocity, speed, and acceleration for planar motion.
- Explain the difference between displacement and distance traveled, and how to compute each.
- Convert between polar and Cartesian coordinates in both directions.
- Write the polar derivative formula and state which derivative's zeros give horizontal vs vertical tangents.
- For $r = 1 + \cos \theta$, find where the horizontal tangents occur and identify the cusp.
- Write the polar area formula and explain where the $1/2$ comes from.
- Explain why the θ -interval must trace a polar region exactly once, with an example.
- Write the area-between-two-polar-curves formula and explain the origin subtlety.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Parametric, Polar, and Vector Functions deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Parametric equations, Eliminating the parameter, Derivative of a parametric function, Second derivative of a parametric function, Arc length of a parametric curve, Vector-valued function, Derivative of a vector-valued function, Integral of a vector-valued function, Position from a velocity vector, Velocity vector, Speed of a particle in planar motion, Acceleration in planar motion, Displacement vs. distance traveled, Polar coordinates, Derivative in polar form, Area of a polar region, Area between two polar curves.

About this guide. Written for Rycal and aligned to the College Board AP Calculus BC course framework, Unit 9. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.