

Unit 8: Applications of Integration

Unit 8 turns the definite integral into a tool for measuring things. It covers average value, displacement and total distance, accumulation in applied settings, area between curves, volumes by cross sections and by revolution, and arc length. Every topic in this unit is the same move: chop a quantity into thin slices, approximate each slice, and add them with an integral.

AP Calculus BC

Applications of Integration

About 12 minutes to read

How to use this guide

Read it in order the first time because the setup pattern repeats. The first three topics (average value, displacement, accumulation) teach you to read what an integral is measuring. The middle topics (area, cross sections, discs, washers) teach you to build the integrand from geometry. Arc length closes the unit with one more integrand to memorize. Exam questions in this unit are almost always setup questions: the points are in choosing the right integrand and limits, not in evaluating.

After the first read, use the trap boxes to review the distinctions the exam tests most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Applications of Integration is one of the heaviest units on the AP Calculus BC exam, and it is where free-response questions live. FRQs in this unit routinely ask you to set up and sometimes evaluate integrals for area, volume, and motion, so practice writing complete setups with correct limits and integrands.

8.1 Average Value of a Function

The **average value of a function** f over the interval $[a, b]$ is the integral of f over the interval divided by the length of the interval: $(1/(b-a))\int_a^b f(x) dx$. Think of it as spreading the total accumulated area evenly across the interval, so the average value is the constant height of a rectangle with the same area.

For example, the average value of $f(x) = x^2$ on $[0, 2]$ is $(1/2)\int_0^2 x^2 dx = (1/2)(8/3) = 4/3$. The integral gives $8/3$, and dividing by the interval length 2 gives $4/3$.

Trap. The factor $1/(b-a)$ is the most dropped piece of this formula. An answer of $8/3$ above is the integral, not the average value. If the question asks for an average, the division by the interval length has to happen.

8.2 Displacement and Total Distance

Displacement is the net change in position: $\int_a^b v(t) dt$, where $v(t)$ is velocity. It can be negative, because motion backward cancels motion forward. **Total distance traveled** is the full path length: $\int_a^b |v(t)| dt$, which is never negative, because every bit of motion counts.

To compute total distance, find where $v(t) = 0$ inside the interval and split the integral there, flipping the sign on each piece so every contribution is positive. For $v(t) = t^2 - 4$ on $[0, 3]$, the velocity is zero at $t = 2$. Displacement is $\int_0^3 (t^2 - 4) dt = -3$. Total distance is $\int_0^2 (4 - t^2) dt + \int_2^3 (t^2 - 4) dt = 16/3 + 7/3 = 23/3$.

Trap. Displacement and total distance are different questions with different answers, as the example shows: -3 versus $23/3$. When a problem asks how far a particle traveled, it wants total distance, which means splitting at the zeros of velocity. Integrating $v(t)$ straight through gives displacement, not distance.

8.3 Net Change and Accumulation

Net change is the Fundamental Theorem restated for applied settings: $\int_a^b f'(x) dx = f(b) - f(a)$. When f' is a rate, its definite integral gives the total change in the quantity over the interval. Water flowing into a tank at $r(t) = 20 - t^2$ liters per minute for $0 \leq t \leq 3$ adds $\int_0^3 (20 - t^2) dt = 51$ liters.

This is the same idea as displacement, generalized. Velocity is the rate of change of position, so integrating velocity gives change in position. Any rate integrates to change in its quantity: a flow rate gives volume change, a growth rate gives population change.

8.4 Area Between Curves (Functions of x)

The **area between curves** $y = f(x)$ and $y = g(x)$ on $[a, b]$, with $f \geq g$ throughout, is $\int_a^b [f(x) - g(x)] dx$. The integrand is always top minus bottom. For $y = x$ and $y = x^2$ on $[0, 1]$, the line is on top, so the area is $\int_0^1 (x - x^2) dx = 1/6$.

When the curves cross inside the interval, find the intersection points and split the integral there, because which curve is on top changes. The limits of integration are usually the x-coordinates of the intersections, not numbers handed to you.

Trap. Top minus bottom is not optional ordering. Integrating (bottom – top) gives the negative of the area. If the curves cross and you integrate straight through without splitting, the signed regions cancel and you get the wrong answer.

8.5 Area Between Curves (Functions of y)

When the region is described more naturally with horizontal slices, integrate with respect to y . The area between $x = f(y)$ on the right and $x = g(y)$ on the left, for $c \leq y \leq d$, is $\int_c^d [f(y) - g(y)] dy$. The integrand is right minus left, mirroring top minus bottom.

For $x = y + 2$ and $x = y^2$, the curves meet where $y + 2 = y^2$, which gives $y = -1$ and $y = 2$. On $[-1, 2]$ the line $x = y + 2$ is to the right, so the area is $\int_{-1}^2 [(y + 2) - y^2] dy = 9/2$.

Trap. The limits here are y-values, and the integrand subtracts left from right. Mixing up the variable is the standard error: writing dx with y-limits, or subtracting top from bottom when the slices are horizontal. Match the differential to the limits.

Note. This guide follows CED topic numbering, so a few numbers are skipped. Topic 8.6 (area between curves that intersect at more than two points), 8.8 (cross sections that are triangles and semicircles), 8.10 (disc method around other axes), and 8.12 (washer method around other axes) are extensions of the methods taught here and have no separate sections.

8.7 Volume with Cross Sections

A solid whose **cross sections** perpendicular to the x -axis have area $A(x)$ has volume $\int_a^b A(x) dx$. Each thin slice contributes its area times its thickness dx , and the integral adds them. For square cross sections with side length $s(x)$, the area is $A(x) = s(x)^2$. For rectangles with length $l(x)$ and width $w(x)$, it is $A(x) = l(x)w(x)$.

Take the region under $y = \sqrt{x}$ from 0 to 4 as the base, with square cross sections perpendicular to the x -axis. The side length is the height of the region, $s(x) = \sqrt{x}$, so $A(x) = (\sqrt{x})^2 = x$, and the volume is $\int_0^4 x dx = 8$.

Trap. The cross-sectional area is a function of x that gets squared (or multiplied) before integrating. Writing $\int s(x) dx$ instead of $\int [s(x)]^2 dx$ is the classic setup error. The side length is not the area.

8.9 Disc Method

Revolving a region around an axis produces discs. The **disc method** gives $V = \pi \int_a^b [R(x)]^2 dx$, where $R(x)$ is the radius from the curve to the axis of revolution. Each disc has area πR^2 and thickness dx .

Revolve $y = \sqrt{x}$ from 0 to 4 around the x -axis. The radius is $R(x) = \sqrt{x}$, so $V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = 8\pi$.

Trap. The π sits outside the integral, and the radius gets squared inside. Both $V = \int \pi [R(x)]^2 dx$ and $V = \pi \int [R(x)]^2 dx$ are correct, but $V = \pi \int R(x) dx$ (radius not squared) and $V = \pi^2 \int [R(x)]^2 dx$ (π squared) are not.

8.11 Washer Method

When the revolved region has a gap between the curve and the axis, each slice is a washer: a disc with a hole. The **washer method** gives $V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$, where $R(x)$ is the outer radius and $r(x)$ is the inner radius. Subtract the hole's area from the disc's area, then integrate.

Revolve the region between $y = x$ and $y = x^2$ on $[0, 1]$ around the x -axis. The outer radius is $R(x) = x$ and the inner radius is $r(x) = x^2$, so $V = \pi \int_0^1 (x^2 - x^4) dx = \pi(1/3 - 1/5) = 2\pi/15$.

Trap. Square each radius before subtracting. Computing $\pi \int [R(x) - r(x)]^2 dx$, the area of the ring's width treated as a disc, is wrong. The washer area is outer disc minus inner disc: $\pi R^2 - \pi r^2$.

8.13 Arc Length

The **arc length** of a smooth curve $y = f(x)$ on $[a, b]$ is $\int_a^b \sqrt{1 + (f'(x))^2} dx$. It also gives the distance a particle travels along the curve. For $y = (2/3)x^{3/2}$ on $[0, 1]$, $f'(x) = x^{1/2}$, so the length is $\int_0^1 \sqrt{1 + x} dx = (2/3)(2\sqrt{2} - 1)$.

Arc length integrals rarely evaluate by hand to anything clean, and the exam knows it. Most arc length questions ask only for the setup, or they give a calculator-active free-response part. Write the integrand correctly and stop worrying about the antiderivative.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Displacement vs total distance	Displacement is $\int v(t) dt$ and can be negative. Total distance is $\int v(t) dt$ and needs the integral split at velocity zeros.
Average value vs integral	The integral is the total. The average value divides by $(b - a)$. They differ by exactly that factor.
Top – bottom vs right – left	Vertical slices use top minus bottom with dx . Horizontal slices use right minus left with dy . Match the differential to the limits.
Disc vs washer	Disc: one radius, $\pi \int R^2 dx$. Washer: two radii, $\pi \int (R^2 - r^2) dx$. A gap between region and axis means washer.
Cross section area vs side length	The area formula (s^2 , lw , πr^2) goes inside the integral. The linear measurement alone does not.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. The average value of $f(x) = 3x^2$ on the interval $[1, 3]$ is
 - A. 13
 - B. 26
 - C. 9
 - D. $27/2$

2. A particle moves with velocity $v(t) = 3t^2 - 12$ for $0 \leq t \leq 3$. The total distance the particle travels is
 - A. -9
 - B. 16
 - C. 23
 - D. 7

3. The region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ is revolved around the x -axis. The volume of the solid is
 - A. $\pi/6$
 - B. $\pi/3$
 - C. $\pi/2$
 - D. $5\pi/6$

4. Which of the following gives the arc length of $y = x^2$ on $[0, 1]$?
 - A. $\int_0^1 \sqrt{1 + 4x^2} \, dx$
 - B. $\int_0^1 \sqrt{1 + 2x} \, dx$
 - C. $\int_0^1 (1 + 4x^2) \, dx$
 - D. $\int_0^1 \sqrt{1 + x^4} \, dx$

Answer Key

- 1. A.** Average value = $(1/(3-1))\int_1^3 3x^2 dx = (1/2)[x^3]_1^3 = (1/2)(27-1) = 13$. B is the integral without dividing by the interval length. C evaluates $[x^3]$ incorrectly as $27-18$. D divides by 2 but uses only the upper limit value 27.
- 2. C.** $v(t) = 0$ at $t = 2$. Split: $\int_0^2 (12-3t^2) dt = 16$ and $\int_2^3 (3t^2-12) dt = 7$, so total distance = 23. A is the displacement (-9), which is what you get integrating straight through. B is only the first piece. D is only the second piece.
- 3. A.** This is a washer with outer radius $\sqrt{x}$ and inner radius x : $V = \pi\int_0^1 ((\sqrt{x})^2 - x^2) dx = \pi\int_0^1 (x - x^2) dx = \pi(1/2 - 1/3) = \pi/6$. B builds a disc from the inner radius only, computing $\pi\int_0^1 x^2 dx = \pi/3$ and forgetting the outer curve entirely. C uses only the outer radius (disc instead of washer). D adds the two terms instead of subtracting.
- 4. A.** $f'(x) = 2x$, so $1 + (f')^2 = 1 + 4x^2$, and the setup is $\int_0^1 \sqrt{1 + 4x^2} dx$. B forgets to square the derivative. C forgets the square root. D squares x^2 itself instead of differentiating first.

One-Page Recall Check

- Write the average value formula and explain what the $1/(b - a)$ factor does.
- State the displacement and total distance formulas and explain when each applies.
- For $v(t) = t^2 - 4$ on $[0, 3]$, find the displacement and the total distance from scratch.
- State the net change theorem and give an applied example with units.
- Write the area-between-curves setup for functions of x and explain top minus bottom.
- Write the area-between-curves setup for functions of y and explain right minus left.
- Write the cross-sectional volume formula and the square and rectangle area rules.
- Write the disc method formula and identify the radius in a worked example.
- Write the washer method formula and explain why each radius is squared before subtracting.
- Write the arc length formula and set it up for $y = x^2$ on $[0, 1]$.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Applications of Integration deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Average value of a function, Displacement, Total distance traveled, Net change, Area between curves (functions of x), Area between curves (functions of y), Volume with cross sections, Disc method, Washer method, Arc length of a planar curve

About this guide. Written for Rycal and aligned to the College Board AP Calculus BC course framework, Unit 8. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.