

Unit 7: Differential Equations

Unit 7 turns derivatives into models. It covers what a differential equation says about a changing quantity, how to check whether a function satisfies one, how slope fields picture every solution at once, how Euler's method approximates a solution step by step, and how to solve by separating variables. It closes with the two models the exam returns to again and again: exponential and logistic growth.

AP Calculus BC

Differential Equations

About 12 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. The equation tells you how a quantity changes, the slope field shows all possible solutions at once, Euler's method walks along one solution numerically, and separation of variables solves the equation exactly. The exponential and logistic models at the end are applications of everything before them.

After the first read, use the trap boxes and the confusions table to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Differential equations appear regularly in the free-response section, often as a multi-part question that moves from a slope field to Euler's method to an exact solution. The separation-of-variables technique also shows up inside later units whenever a rate equation needs solving.

7.1 Modeling Situations with Differential Equations

A **differential equation** is a relation that ties a function to its derivatives. The equation $dy/dx = ky$ does not give you y directly. It tells you how y changes: at every point, the rate of change of y is proportional to y itself. When a quantity grows or shrinks in proportion to its current size, like a population or a cooling object, the situation is modeled by an equation of this form.

Reading the equation is a skill worth practicing. The left side names the rate, dy/dx or dy/dt . The right side names the rule the rate follows. In $dy/dx = 2xy$, the rate depends on both x and y . In $dy/dt = ky(a - y)$, the rate depends on y and on how far y is from the value a . Before solving anything, make sure you can say in words what the right side claims about the quantity.

Trap. The differential equation is not the solution. The equation $dy/dx = 2x$ describes how y changes. The function $y = x^2 + 3$ is a solution. Exam questions sometimes ask which of two statements is the equation and which is the solution, so keep the roles straight.

7.2 Verifying Solutions for Differential Equations

A **solution of a differential equation** is a function that satisfies the equation, along with its derivatives, when substituted in. To verify a proposed solution, compute its derivative, plug both the function and the derivative into the equation, and check that the left side equals the right side.

Check $y = 5e^{x^2}$ against $dy/dx = 2xy$. The derivative is $y' = 5e^{x^2} \cdot 2x$, by the chain rule. The right side is $2x \cdot y = 2x \cdot 5e^{x^2}$. Both sides equal $10xe^{x^2}$, so the function is a solution. The check has two parts: differentiate correctly, then confirm the substitution balances.

Trap. A function that satisfies the equation at one point is not automatically a solution. Verification means the equation holds for every x in the domain, because you substitute the whole function and its whole derivative, not a single value.

7.3 Sketching Slope Fields

A **slope field** is a grid of short line segments showing the value of dy/dx at many points. At each point (x, y) , you evaluate the right side of the differential equation and draw a tiny segment with that slope. A solution curve is any curve that follows the segments, staying tangent to them as it passes through.

To sketch one by hand, pick a few y values and compute the slope at each. For $dy/dx = x + y$, at $(0, 0)$ the slope is 0, at $(0, 1)$ the slope is 1, at $(1, 0)$ the slope is 1, and at $(-1, 1)$ the slope is 0. Draw short segments with those slopes at those points. Where the right side is zero, the segments are horizontal, and those points mark where solution curves flatten out.

7.4 Reasoning Using Slope Fields

Reading a slope field is just as important as drawing one. If the segments get steeper as y increases, solutions grow faster at larger y . If the segments are horizontal along some line $y = c$, that line may be a constant solution. When a question gives you a slope field and asks which differential equation it matches, test a point: compute the right side at that point and check it against the segment you see.

Trap. A slope field shows the slopes of solutions, not the solutions themselves. The segments are tangent lines, not pieces of the curve. A solution curve passes through the grid following the segments, but it is not drawn as a connect-the-dots path through the segment endpoints.

7.5 Approximating Solutions Using Euler's Method

Euler's method approximates a solution curve numerically. Start from a known point, compute the slope there from the differential equation, and step forward along that slope to a new point. Then repeat: compute the new slope, step forward again. Each step uses the formula $y_{\text{new}} = y + (\Delta x)(dy/dx)$, evaluated at the current point.

Work it through for $dy/dx = x + y$ with $y(0) = 1$, approximating $y(0.2)$ with step size $\Delta x = 0.1$. Start at $(0, 1)$. The slope is $0 + 1 = 1$, so $y_1 = 1 + 0.1(1) = 1.1$, and $x_1 = 0.1$. At $(0.1, 1.1)$, the slope is $0.1 + 1.1 = 1.2$, so $y_2 = 1.1 + 0.1(1.2) = 1.1 + 0.12 = 1.22$, and $x_2 = 0.2$. The approximation is $y(0.2) \approx 1.22$.

Notice what each step needs: the current x , the current y , and the slope from the equation at that point. Smaller step sizes give better approximations but require more steps. On the exam, show every step in a table or a list so the reader can follow the arithmetic.

Trap. The most common error is reusing the old slope instead of recomputing it at the new point. After the first step you are at $(0.1, 1.1)$, not $(0, 1)$, so the slope for the second step is 1.2, not 1. Update both x and y before computing the next slope.

7.6 Finding General Solutions Using Separation of Variables

Separation of variables solves equations of the form $dy/dx = g(x) \cdot h(y)$. Rewrite the equation so that all the y terms are on one side with dy and all the x terms are on the other side with dx : $(1/h(y)) dy = g(x) dx$. Then integrate both sides.

Solve $dy/dx = 2xy$. Divide both sides by y to get $(1/y) dy = 2x dx$. Integrate: the left side gives $\ln|y|$, and the right side gives $x^2 + C$. So $\ln|y| = x^2 + C$. Exponentiate both sides: $|y| = e^{x^2 + C} = e^C \cdot e^{x^2}$. Since e^C is a positive constant, absorbing the sign gives $y = Ae^{x^2}$, where A is an arbitrary constant.

The **general solution** is the whole family of functions satisfying the equation, parameterized by the arbitrary constant C (or A). Every choice of the constant gives a different member of the family. The absolute value in $\ln|y|$ is handled by letting the constant absorb the sign, which is why the final form drops the bars.

Trap. Do not forget the constant of integration, and do not add it twice. One constant is enough because combining C_1 and C_2 from the two sides into a single C covers every possibility. Also, the constant goes in as soon as you integrate, not at the end.

7.7 Finding Particular Solutions Using Initial Conditions

An **initial condition** is a specified solution value, such as $y(0) = 3$. It picks one member out of the general solution's family. A **particular solution** is the single function satisfying both the equation and the initial condition, with no arbitrary constant remaining.

Finish the example from 7.6: $dy/dx = 2xy$ with the initial condition $y(0) = 5$. The general solution is $y = Ae^{x^2}$. Substitute $x = 0$ and $y = 5$: $5 = Ae^0 = A$, so $A = 5$. The particular solution is $y = 5e^{x^2}$. As a check, $y' = 5e^{x^2} \cdot 2x = 2x \cdot y$, which matches the original equation, and $y(0) = 5e^0 = 5$, which matches the condition.

There are two places to apply the condition: after solving for y in terms of the constant, or right after integrating by using the condition to evaluate a definite integral. Either way works. The key step is substituting the known x and y values and solving for the constant before writing the final answer.

Trap. A particular solution has no arbitrary constant left in it. If your final answer still contains C or A , you have written the general solution, not the particular one. The initial condition exists precisely to determine that constant.

7.8 Exponential Models with Differential Equations

The **exponential model** is the equation $dy/dt = ky$. It says the rate is proportional to the quantity itself. The solutions are $y = Ce^{kt}$: they grow when k is greater than 0 and decay when k is less than 0. This is the same family you met in 7.6, now read as a model of real quantities.

Solve $dy/dt = 3y$ with $y(0) = 4$. Separate: $(1/y) dy = 3 dt$. Integrate: $\ln|y| = 3t + C$, so $y = Ae^{3t}$. The initial condition gives $4 = Ae^0$, so $A = 4$. The particular solution is $y = 4e^{3t}$. Checking: $y' = 12e^{3t}$ and $3y = 12e^{3t}$, and $y(0) = 4$. The constant $k = 3$ is the relative growth rate: the quantity grows at 3 per unit of itself per unit time.

Exponential models fit situations with unlimited growth or decay: unrestricted populations, continuously compounded interest, radioactive decay. The sign of k tells the whole story. Positive k means the quantity increases without bound, negative k means it decreases toward zero.

Trap. In $y = Ce^{kt}$, the constant C is the initial value $y(0)$, not a free parameter to ignore. And k is not the growth itself, it is the proportional rate. A question asking for the population after 5 years needs both numbers substituted, not just the form of the solution.

7.9 Logistic Models with Differential Equations

The **logistic growth model** is $dy/dt = ky(a - y)$. The rate is jointly proportional to the current size y and the remaining room $(a - y)$. Early on, when y is small, $(a - y)$ is nearly a , and growth looks exponential. As y approaches a , the remaining room shrinks and growth slows to a stop.

The **carrying capacity** is the limiting value a . As the independent variable goes to infinity, solutions approach a without crossing it. The **point of fastest logistic growth** is the inflection point of the solution curve, at $y = a/2$, where the rate is greatest. You can find it without solving the equation: it is the y value that maximizes the product $y(a - y)$.

For $dy/dt = 0.1y(100 - y)$, the carrying capacity is 100. The fastest growth happens at $y = 50$, where the rate is $0.1 \cdot 50 \cdot 50 = 250$. At $y = 25$ the rate is $0.1 \cdot 25 \cdot 75 = 187.5$, and at $y = 75$ it is $0.1 \cdot 75 \cdot 25 = 187.5$, confirming that the rate peaks in the middle and falls off symmetrically on both sides.

Trap. The carrying capacity is a horizontal asymptote of the solution, not a value the solution reaches. A logistic curve approaches a as time goes to infinity but never touches it. Also, the fastest growth is at half the carrying capacity, not at the carrying capacity itself, where the rate is zero.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
General vs particular solution	The general solution keeps the arbitrary constant and describes the whole family. The particular solution uses the initial condition to fix the constant and names one function.
Slope field vs solution curve	The field is the grid of tangent segments. A solution curve is one path through the grid, following the segments. The field contains every solution at once.
Euler's method vs exact solution	Euler's method approximates by walking along tangent lines in steps. Separation of variables gives the exact formula. Euler is for when you cannot or need not solve exactly.
Exponential vs logistic model	Exponential, $dy/dt = ky$, grows without bound. Logistic, $dy/dt = ky(a - y)$, levels off at the carrying capacity a . The $(a - y)$ factor is what stops the growth.
Carrying capacity vs fastest growth point	The carrying capacity a is the ceiling the solution approaches. The fastest growth happens at $y = a/2$, the inflection point, where the rate is greatest.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Let $dy/dx = x + y$ and $y(0) = 1$. Using Euler's method with step size $\Delta x = 0.1$, what is the approximation for $y(0.2)$?

- A. 1.10
- B. 1.20
- C. 1.22
- D. 1.24

2. Which of the following is the general solution of $dy/dx = 2xy$?

- A. $y = e^{x^2} + C$
- B. $y = Ae^{x^2}$
- C. $y = 2x^2 + C$
- D. $y = Ae^{2x}$

3. A population P satisfies $dP/dt = 0.2P(500 - P)$. What is the carrying capacity, and at what population size is the growth rate greatest?

- A. Carrying capacity 500; fastest growth at $P = 500$
- B. Carrying capacity 500; fastest growth at $P = 250$
- C. Carrying capacity 250; fastest growth at $P = 250$
- D. Carrying capacity 0.2; fastest growth at $P = 100$

4. If $dy/dt = 3y$ and $y(0) = 4$, then $y(1) =$

- A. $4e^3$
- B. 12
- C. $3e^4$
- D. $4 + 3e$

Answer Key

1. C. Step 1: at $(0, 1)$, the slope is $0 + 1 = 1$, so $y_1 = 1 + 0.1(1) = 1.1$ and $x_1 = 0.1$. Step 2: at $(0.1, 1.1)$, the slope is $0.1 + 1.1 = 1.2$, so $y_2 = 1.1 + 0.1(1.2) = 1.22$. A stops after one step. B comes from reusing the old slope for both steps, computing $1 + 0.1(1) + 0.1(1) = 1.20$ instead of updating the slope to 1.2 at the second step. D may come from using the wrong slope at the second step.

2. B. Separate: $(1/y) dy = 2x dx$. Integrate to get $\ln|y| = x^2 + C$, then exponentiate: $y = Ae^{x^2}$. A adds the constant outside the exponential, which does not come from integrating correctly. C integrates the right side as if it were the whole equation, ignoring the separation step. D treats $2x$ as a constant multiplier, giving the wrong exponent.

3. B. In the logistic form $dy/dt = ky(a - y)$, the carrying capacity is $a = 500$, and the fastest growth is at $y = a/2 = 250$. A confuses the fastest growth point with the carrying capacity itself, where the rate is actually zero. C halves the carrying capacity by mistake. D reads the rate constant 0.2 as the capacity.

4. A. The equation $dy/dt = 3y$ has solutions $y = Ce^{3t}$. The initial condition gives $4 = Ce^0$, so $C = 4$ and $y = 4e^{3t}$. At $t = 1$, $y(1) = 4e^3$. B multiplies the rate constant by the initial value, which is the derivative at $t = 0$, not the value at $t = 1$. C swaps the roles of the constant and the rate. D adds terms instead of using the exponential form.

One-Page Recall Check

- Explain what a differential equation models and give an example in words.
- Verify that $y = 5e^{x^2}$ satisfies $dy/dx = 2xy$, showing both steps.
- Describe how to read a slope field and how to sketch one for $dy/dx = x + y$.
- State the Euler's method step formula and explain why the slope must be recomputed each step.
- Use Euler's method with $\Delta x = 0.1$ to approximate $y(0.2)$ for $dy/dx = x + y$, $y(0) = 1$.
- Solve $dy/dx = 2xy$ by separation of variables, showing the integration and the constant.
- Explain the difference between a general solution and a particular solution.
- Apply $y(0) = 5$ to $y = Ae^{x^2}$ and write the particular solution.
- Solve $dy/dt = 3y$ with $y(0) = 4$ and verify the answer by substitution.
- State the logistic equation and identify the carrying capacity in $dy/dt = 0.1y(100 - y)$.
- Explain why the fastest logistic growth happens at $y = a/2$.
- Give the rate of change at $y = 25$, $y = 50$, and $y = 75$ for $dy/dt = 0.1y(100 - y)$.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Differential Equations deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Differential equation, Solution of a differential equation, Slope field, Euler's method, Separation of variables, General solution, Particular solution, Initial condition, Exponential model, Logistic growth model, Carrying capacity, Point of fastest logistic growth.

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