

Unit 6: Integration and Accumulation of Change

Unit 6 is where integration stops being an area problem and becomes the general tool for undoing a rate. It starts with accumulation, builds the definite integral out of Riemann sums, states the Fundamental Theorem, and then adds the BC-only integration techniques: substitution, long division, integration by parts, partial fractions, and improper integrals.

AP Calculus BC

Integration and Accumulation of Change

About 14 minutes to read

How to use this guide

Read it in order the first time because the techniques build on each other. Accumulation motivates the integral, Riemann sums define it, the Fundamental Theorem connects it to antiderivatives, and then each integration method handles a family of integrands the basic rules cannot touch. The BC-only methods (long division, parts, partial fractions, improper integrals) are the ones the exam tests most heavily in this unit.

After the first read, use the trap boxes and the technique-selection table to practice choosing the right method. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Integration and Accumulation of Change is about 17 to 20 percent of the AP Calculus BC exam, the largest single-unit share. The techniques here also feed directly into Unit 7 (differential equations), Unit 8 (applications of integration), and Unit 10 (series), where integrals appear as error bounds and convergence tests.

6.1 Exploring Accumulations of Change

The **accumulation of change** is the net total change of a quantity over an interval, found by integrating its rate of change. If water flows into a tank at rate $r(t)$ gallons per minute, then the total gallons added between $t = a$ and $t = b$ is the integral of r from a to b . This is the idea underneath every applied integral: a rate in, an integral out.

Accumulation is net, not total. If the rate goes negative, those contributions subtract. A tank that fills at 5 gal/min for 3 minutes and drains at 2 gal/min for 3 minutes accumulates $15 - 6 = 9$ gallons. The integral records the signed sum, which is why accumulation can be less than the total distance traveled or total amount moved.

Trap. Accumulation is the net change, not the total variation. If a rate changes sign on the interval, the integral subtracts the negative part. A question that asks for total distance or total amount needs the integral of the absolute value, not the plain integral.

6.2 Approximating Areas with Riemann Sums

A **Riemann sum** approximates area by summing rectangle areas: $\sum f(x_i^*) \cdot \Delta x$ over the subintervals, where x_i^* is a sample point in each subinterval. The choice of sample point gives the sum its name. A **left Riemann sum** uses the function value at the left endpoint of each subinterval, a **right Riemann sum** uses the right endpoint, and a **midpoint Riemann sum** uses the midpoint.

A **trapezoidal sum** averages the left and right heights instead of picking one: each subinterval contributes $(\Delta x/2) \cdot [f(x_{i-1}) + f(x_i)]$. Geometrically it uses trapezoids rather than rectangles. For $f(x) = x^2$ on $[0, 2]$ with $n = 4$, $\Delta x = 0.5$: the right sum is $0.5 \cdot (0.25 + 1 + 2.25 + 4) = 3.75$, the midpoint sum is $0.5 \cdot (0.0625 + 0.5625 + 1.5625 + 3.0625) = 2.625$, and the trapezoidal sum is $0.25 \cdot (0 + 0.5 + 2 + 4.5 + 4) = 2.75$. The exact value is $8/3 \approx 2.667$, so midpoint wins here.

Trap. For an increasing function, the left sum underestimates and the right sum overestimates; for a decreasing function it flips. Midpoint and trapezoidal are usually closer, but their error directions depend on concavity, not monotonicity. Do not guess the direction from the name alone.

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

The **definite integral** is the exact signed area under f from a to b , defined as the limit of the rectangle approximations: $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum f(x_i^*) \cdot \Delta x$. The Riemann sums are the approximations; the definite integral is what they converge to as the subintervals shrink to zero width.

Summation (sigma) notation is the compact symbol for writing sums: $\sum_{i=1}^n a_i$ means $a_1 + a_2 + \dots + a_n$. It is the bookkeeping language of Riemann sums. Being fluent in it matters because the definition of the integral is stated as a limit of a sigma expression, and series in Unit 10 use the same notation.

Trap. The definite integral is a number, not a function. $\int_a^b f(x) dx$ evaluates to a single value once the limits are fixed. If the upper limit is a variable, that is an accumulation function, which is a different object with different rules.

6.4 The Fundamental Theorem of Calculus and Accumulation Functions

The **Fundamental Theorem of Calculus** says differentiation and integration are inverse processes, in two parts. First, if $F(x) = \int_a^x f(t) dt$ for continuous f , then $F'(x) = f(x)$. Second, if f is continuous on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$ for any antiderivative F . The first part lets you differentiate an integral; the second lets you evaluate one.

An **accumulation function** is $A(x) = \int_a^x f(t) dt$, the running net area under f from a to x . Its graph starts at zero when $x = a$, rises where f is positive, and falls where f is negative. Reading an accumulation function off the graph of f is a standard exam task: the value at x is the signed area accumulated so far.

Trap. When differentiating an accumulation function with a non- x upper limit, apply the Chain Rule. If $G(x) = \int_a^{x^2} f(t) dt$, then $G'(x) = f(x^2) \cdot 2x$, not just $f(x^2)$. The derivative of the upper limit multiplies the result.

6.5 Interpreting the Behavior of Accumulation Functions Involving Area

The behavior of an accumulation function $A(x) = \int_a^x f(t) dt$ is read directly from f . Where f is above the x -axis, A is increasing; where f is below, A is decreasing. Where f crosses zero, A has a relative extremum. Where f has a max or min and f' changes sign there, A has an inflection point, because $A'' = f'$.

This is the graphical version of the first part of the Fundamental Theorem. Since $A'(x) = f(x)$, every fact about derivatives applies: critical points of A are zeros of f , and the sign chart of f is the increasing/decreasing chart of A . Exam questions often give the graph of f and ask for the graph of A , or ask where A attains its maximum on an interval.

6.6 Applying Properties of Definite Integrals

The **properties of definite integrals** let you split, scale, and reorder: $\int_a^b [f \pm g] = \int_a^b f \pm \int_a^b g$; $\int_a^b c \cdot f = c \cdot \int_a^b f$; $\int_a^b f = -\int_b^a f$; $\int_a^a f = 0$; and $\int_a^b f + \int_b^c f = \int_a^c f$. These are the algebraic moves that justify breaking a hard integral into manageable pieces.

The interval-addition property is the workhorse: if you know the integral over $[a, b]$ and over $[b, c]$, you know it over $[a, c]$. Combined with the reversal property, it lets you rearrange any collection of definite integrals over adjacent intervals into whatever form the question needs.

6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

An **antiderivative** is a function F with $F'(x) = f(x)$, and any two antiderivatives of the same function differ by a constant. The **indefinite integral** is the family of all antiderivatives, written $\int f(x) dx = F(x) + C$, where C is the constant of integration. Every indefinite integral answer needs the $+ C$; without it the answer is incomplete.

The basic rules are the derivative rules run backward: power rule, exponentials, trig functions, $1/x$. These handle the straightforward integrands. Everything from 6.9 onward exists because most integrands are not in basic form and need to be rewritten first.

Trap. The $+ C$ is required on every indefinite integral, and it is a single constant even after multiple steps. A question that asks you to "find the antiderivative" with no limits wants $F(x) + C$, not a number.

6.9 Integrating Using Substitution

Integration by substitution is the reverse Chain Rule: with $u = g(x)$, $\int f(g(x)) \cdot g'(x) dx = \int f(u) du$. Pick u to be the inner function whose derivative also appears in the integrand. For $\int 2x \cdot \cos(x^2) dx$, $u = x^2$ gives $du = 2x dx$, and the integral becomes $\int \cos(u) du = \sin(u) + C = \sin(x^2) + C$.

For definite integrals, change the limits when you substitute rather than switching back: if $u = x^2$ and x runs from 0 to 1, then u runs from 0 to 1 as well, and $\int_0^1 2x \cdot \cos(x^2) dx = \int_0^1 \cos(u) du = \sin(1) - \sin(0) = \sin(1)$. Changing the limits avoids the round trip back to x .

Trap. Substitution only works when the derivative of your chosen u is present (up to a constant factor). If you pick $u = x^2$ but there is no x factor in the integrand, the substitution goes nowhere. Check for the du before committing.

6.10 Integrating Using Long Division and Completing the Square

Integrating with long division and completing the square handles rational integrands that are not yet in integrable form. When the numerator's degree is at least the denominator's, divide first: the quotient integrates as a polynomial and the remainder becomes a proper fraction. For $\int (x^2 + 1)/(x + 1) dx$, long division gives $x - 1$ with remainder 2, so the integral is $\int (x - 1) dx + \int 2/(x + 1) dx = x^2/2 - x + 2 \ln|x + 1| + C$.

Completing the square rewrites a quadratic denominator into a sum of squares, which integrates to an arctangent: $x^2 + 2x + 5 = (x + 1)^2 + 4$, so $\int dx/(x^2 + 2x + 5) = \int du/(u^2 + 4) = (1/2) \arctan(u/2) + C = (1/2) \arctan((x + 1)/2) + C$. Recognize $u^2 + a^2$ in the denominator and think arctangent.

Trap. Always check the degrees before choosing a method. If the numerator degree is greater than or equal to the denominator degree, divide first. Partial fractions require a proper fraction, so skipping the division step guarantees a wrong setup.

6.11 Integrating Using Integration by Parts

Integration by parts is $\int u dv = uv - \int v du$. Pick u as the part that simplifies when differentiated and dv as the part you can integrate. For $\int x \cdot e^x dx$, $u = x$ and $dv = e^x dx$ give $du = dx$ and $v = e^x$, so the integral is $x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + C = e^x(x - 1) + C$. Check by differentiating: $d/dx[e^x(x - 1)] = e^x(x - 1) + e^x = x \cdot e^x$.

The standard priority for choosing u runs: inverse trig, logarithmic, algebraic (polynomial), trigonometric, exponential. Logarithms and inverse trig almost always become u because they simplify dramatically under differentiation while being hard to integrate directly. Some integrals need parts twice, and a few cycle back to the original integral, which you then solve for algebraically.

Trap. The most common error is picking u and dv backwards, which makes the new integral harder instead of easier. If $\int v du$ looks worse than what you started with, swap your choice. Also, do not forget that dv must include the dx .

6.12 Integrating Using Linear Partial Fractions

Partial fractions (linear, nonrepeating factors) split a proper rational function whose denominator factors into distinct linear pieces: rewrite it as a sum of simple $A/(x - a) + B/(x - b)$ terms and integrate each to a logarithm. For $\int (x + 3)/(x^2 - 1) dx$, the denominator is $(x - 1)(x + 1)$, so write $(x + 3)/(x^2 - 1) = A/(x - 1) + B/(x + 1)$. Then $x + 3 = A(x + 1) + B(x - 1)$: setting $x = 1$ gives $4 = 2A$, so $A = 2$; setting $x = -1$ gives $2 = -2B$, so $B = -1$. The integral is $2 \ln|x - 1| - \ln|x + 1| + C$.

Verify the split at a test value before integrating: at $x = 0$, the original is $(0 + 3)/(0 - 1) = -3$, and $2/(-1) - 1/(1) = -3$. If the test value disagrees, the A and B are wrong and everything downstream is wrong. The method extends to repeated and irreducible quadratic factors, but the BC exam centers on the distinct-linear case.

Trap. Partial fractions require a proper fraction with a factored denominator. If the fraction is improper, do long division first. If the denominator is not factored, factor it first. Both skipped steps are among the most common setup errors.

6.13 Evaluating Improper Integrals

An **improper integral** has an infinite limit of integration, or an integrand that is unbounded somewhere on the interval. **Convergence of an improper integral** is decided by evaluating it as a limit of ordinary definite integrals: it converges if that limit is a finite number, and diverges if the limit is infinite or does not exist.

For $\int_1^\infty dx/x^2$, write $\lim(b \rightarrow \infty) \int_1^b x^{-2} dx = \lim(b \rightarrow \infty) [-1/x]_1^b = \lim(b \rightarrow \infty) (-1/b + 1) = 1$, so it converges to 1. For $\int_1^\infty dx/x$, the same process gives $\lim(b \rightarrow \infty) \ln b = \infty$, so it diverges. The exponent makes all the difference: $1/x^2$ converges, $1/x$ does not.

Trap. Never plug ∞ in as if it were a number. An improper integral is always a limit, and you must write the limit step. Also, if the integrand is unbounded at an interior point, split the integral there and take two separate limits; a single limit across the bad point is not valid.

Choosing the Right Technique

WHAT THE INTEGRAND LOOKS LIKE	TRY FIRST
A composition with the inner derivative present	Substitution (6.9)
Numerator degree $\geq$ denominator degree (rational)	Long division, then whatever the remainder needs (6.10)
Product of unrelated function types (polynomial $\times$ exponential, polynomial $\times$ trig, log $\times$ anything)	Integration by parts (6.11)
Proper rational function with factorable denominator	Partial fractions (6.12)
Quadratic denominator that is a sum of squares	Complete the square $\rightarrow$ arctangent (6.10)
Infinite limit or vertical asymptote in the interval	Improper integral: rewrite as a limit (6.13)

Trap. Technique selection is half the battle on free response. Before computing anything, classify the integrand using the table above. The exam rewards the correct setup even when the arithmetic is long, and it punishes the wrong method no matter how carefully executed.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Definite vs indefinite integral	Definite has limits and evaluates to a number. Indefinite has no limits, needs $+ C$, and gives a family of functions.
Left vs right Riemann sum (increasing f)	Left underestimates, right overestimates. For decreasing f it reverses. Midpoint and trapezoidal depend on concavity.
Accumulation function vs antiderivative	An accumulation function is one specific antiderivative, pinned down by $A(a) = 0$. A general antiderivative still has the $+ C$.
u -substitution vs integration by parts	Substitution reverses the Chain Rule and needs the inner derivative present. Parts reverses the Product Rule and handles products of unrelated types.
Long division vs partial fractions	Division comes first and only when the fraction is improper. Partial fractions need a proper fraction with a factored denominator.
Convergent vs divergent improper integral	Convergent means the limit is a finite number. Divergent means the limit is infinite or does not exist. Always show the limit step.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Let f be increasing on $[0, 2]$. Using $n = 4$ subintervals, which approximation of $\int_0^2 f(x) \, dx$ is guaranteed to be an underestimate?

- A. Left Riemann sum
- B. Right Riemann sum
- C. Trapezoidal sum
- D. Midpoint Riemann sum

2. $\int x \cdot \cos(x^2) \, dx =$

- A. $\sin(x^2) + C$
- B. $(1/2) \sin(x^2) + C$
- C. $x \cdot \sin(x^2) + C$
- D. $-\sin(x^2) + C$

3. $\int (2x + 1)/(x^2 - 4) \, dx =$

- A. $(5/4) \ln|x - 2| + (3/4) \ln|x + 2| + C$
- B. $2 \ln|x - 2| - \ln|x + 2| + C$
- C. $\ln|x^2 - 4| + C$
- D. $(1/2) \ln|x^2 - 4| + C$

4. $\int_2^\infty dx/x^3 =$

- A. $1/8$
- B. $1/4$
- C. Diverges
- D. $1/2$

Answer Key

- 1. A.** For an increasing function, every left-endpoint rectangle sits below the curve, so the left sum underestimates. B overestimates for the same reason. C and D depend on concavity: the trapezoidal sum overestimates when f is concave up and underestimates when concave down, and the midpoint sum does the reverse, so neither is guaranteed without knowing the concavity.
- 2. B.** Substitute $u = x^2$, $du = 2x \, dx$, so $x \, dx = du/2$. The integral becomes $(1/2) \int \cos(u) \, du = (1/2) \sin(u) + C = (1/2) \sin(x^2) + C$. A forgets the $1/2$ from the du conversion. C differentiates nothing and just tacks on the antiderivative incorrectly. D has the wrong sign.
- 3. A.** Factor: $x^2 - 4 = (x - 2)(x + 2)$. Write $(2x + 1)/(x^2 - 4) = A/(x - 2) + B/(x + 2)$, so $2x + 1 = A(x + 2) + B(x - 2)$. At $x = 2$: $5 = 4A$, so $A = 5/4$. At $x = -2$: $-3 = -4B$, so $B = 3/4$. The integral is $(5/4) \ln|x - 2| + (3/4) \ln|x + 2| + C$. B uses coefficients from a different numerator. C and D treat the numerator as if it were the derivative of the denominator, but $d/dx[x^2 - 4] = 2x$, not $2x + 1$, so no logarithm shortcut applies.
- 4. A.** Write it as a limit: $\lim_{b \rightarrow \infty} \int_2^b x^{-3} \, dx = \lim_{b \rightarrow \infty} [-1/(2x^2)]_2^b = \lim_{b \rightarrow \infty} (-1/(2b^2) + 1/8) = 1/8$. It converges. C is wrong because the exponent 3 exceeds 1. B and D come from arithmetic slips in evaluating $-1/(2x^2)$ at $x = 2$.

One-Page Recall Check

- Explain what accumulation of change measures and why it is net rather than total.
- Define a Riemann sum and name what distinguishes the left, right, midpoint, and trapezoidal versions.
- For $f(x) = x^2$ on $[0, 2]$ with $n = 4$, compute the right Riemann sum from scratch.
- State the definite integral as a limit of Riemann sums.
- State both parts of the Fundamental Theorem of Calculus.
- Given the graph of f , sketch the accumulation function $A(x) = \int_a^x f(t) dt$ and mark where A has its maximum.
- Differentiate $G(x) = \int_0^{x^2} \sin(t) dt$, showing the Chain Rule step.
- List the five properties of definite integrals.
- Evaluate $\int (x^2 + 1)/(x + 1) dx$ using long division.
- Evaluate $\int x \cdot e^x dx$ by parts, naming your u and dv .
- Evaluate $\int (x + 3)/(x^2 - 1) dx$ by partial fractions, showing the A and B solve.
- Evaluate $\int_1^\infty dx/x^2$ as a limit and state whether it converges.
- Explain why $\int_1^\infty dx/x$ diverges while $\int_1^\infty dx/x^2$ converges.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Integration and Accumulation of Change deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Accumulation of change, Riemann sum, Left Riemann sum, Right Riemann sum, Midpoint Riemann sum, Trapezoidal sum, Definite integral, Summation (sigma) notation, Fundamental Theorem of Calculus, Accumulation function, Properties of definite integrals, Antiderivative, Indefinite integral, Integration by substitution, Integrating with long division and completing the square, Integration by parts, Partial fractions (linear, nonrepeating factors), Improper integral, Convergence of an improper integral

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