

Unit 5: Analytical Applications of Differentiation

Unit 5 turns differentiation into a tool for analyzing functions. It covers the Mean Value Theorem, the Extreme Value Theorem, critical points, the First and Second Derivative Tests, concavity and points of inflection, and optimization problems.

AP Calculus BC

Analytical Applications of Differentiation

About 10 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. The Mean Value Theorem and Extreme Value Theorem set the theoretical guarantees, critical points give you the candidates, the derivative tests classify them, concavity describes the shape, and optimization puts the whole machinery to work. Exam questions reward careful hypothesis checking, so learn each theorem with its conditions attached.

After the first read, use the trap boxes to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Analytical Applications of Differentiation is about 10 to 12 percent of the AP Calculus BC exam. It also matters beyond its own weight, because the habit of checking conditions and justifying with the derivative tests carries into every free-response question that asks you to analyze a function.

5.1 Using the Mean Value Theorem

The **Mean Value Theorem** connects an average rate of change to an instantaneous one. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one c in (a, b) with $f'(c) = [f(b) - f(a)] / (b - a)$. In words, the secant slope between the endpoints equals the tangent slope somewhere in between.

A typical justification: $f(x) = x^2$ is a polynomial, so it is continuous on $[1, 3]$ and differentiable on $(1, 3)$. The average rate over $[1, 3]$ is $(9 - 1) / (3 - 1) = 4$, and $f'(x) = 2x$, so $2c = 4$ gives $c = 2$, which lies in $(1, 3)$. The MVT guarantees such a c exists but never tells you where to look; the algebra finds it.

Trap. The MVT needs both hypotheses. If f is not continuous on $[a, b]$ or not differentiable on (a, b) , the conclusion does not follow. Also, the c the theorem promises is in the open interval (a, b) , never at an endpoint.

5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

The **Extreme Value Theorem** says a function continuous on the closed interval $[a, b]$ attains both an absolute maximum and an absolute minimum there. The closed interval and the continuity are both required. Drop either one and the guarantee disappears: $f(x) = 1/x$ on $(0, 1]$ has no maximum, and a function with a jump on $[0, 1]$ can skip values entirely.

A **local extremum** is a value larger (maximum) or smaller (minimum) than all nearby function values. An **absolute extremum** is the largest or smallest value over the whole domain or a given interval. A **critical point** is a point c in the domain where $f'(c) = 0$ or $f'(c)$ is undefined. Local extrema can only occur at critical points, but a critical point is not automatically an extremum.

Trap. The EVT promises that absolute extrema exist, not where they are. And a critical point where f' does not change sign is not an extremum at all. For $f(x) = x^3$, $x = 0$ is a critical point, but the function is increasing through it, so nothing extreme happens there.

5.3 Determining Intervals on Which a Function is Increasing or Decreasing

Where $f' > 0$, f is increasing, and where $f' < 0$, f is decreasing, provided f is differentiable on the interval in question. The procedure is to find the critical points, split the domain into intervals at those points, and test the sign of f' on each interval. For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$, so the critical points are $x = \pm 1$. Testing gives $f' > 0$ on $(-\infty, -1)$ and $(1, \infty)$, and $f' < 0$ on $(-1, 1)$. The function rises, falls, then rises again.

Trap. State intervals with the correct inclusion. The function is increasing on $(-\infty, -1]$, not just $(-\infty, -1)$: including the endpoint $x = -1$ is fine because the function is continuous there and the monotonicity carries through the point.

5.4 Using the First Derivative Test to Determine Relative (Local) Extrema

The **First Derivative Test** classifies a critical point by the sign change of f' around it. If f' changes from positive to negative at c , f has a local maximum at c . If f' changes from negative to positive, f has a local minimum. If the sign does not change, there is no local extremum. This test works even when $f'(c)$ is undefined rather than zero, which makes it the right choice at corners and cusps where the second derivative test cannot be applied.

For $f'(x) = x(x - 2)^2$, the critical points are $x = 0$ and $x = 2$. For $x < 0$, $f' < 0$; between 0 and 2, $f' > 0$; past 2, $f' > 0$. At $x = 0$ the sign goes from negative to positive, so f has a local minimum there. At $x = 2$ the sign stays positive on both sides, so there is no local extremum even though $x = 2$ is a critical point.

Trap. A critical point with no sign change is not an extremum. Questions love to plant one, like $x = 2$ above, to check whether you actually drew the sign chart or just assumed every critical point qualifies.

5.5 Using the Candidates Test to Determine Absolute (Global) Extrema

The **Candidates Test** finds absolute extrema for a continuous function on a closed interval $[a, b]$. List every interior critical point, evaluate f at each one and at both endpoints, then compare. The largest value is the absolute maximum and the smallest is the absolute minimum. Nothing else needs checking, because the EVT guarantees the extrema exist and they must occur either at a critical point or at an endpoint.

For $f(x) = x^3 - 3x$ on $[-2, 2]$, the critical points are $x = \pm 1$. Evaluating: $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, $f(2) = 2$. The absolute maximum is 2, attained at $x = -1$ and $x = 2$; the absolute minimum is -2 , attained at $x = -2$ and $x = 1$.

Trap. The Candidates Test needs a closed interval and a continuous function. On an open interval or an unbounded domain, the maximum may not exist, and you need a different argument, such as showing the function approaches a value asymptotically or grows without bound.

5.6 Determining Concavity of Functions over Their Domains

Concavity describes the bend of a graph. Where $f''(x) > 0$, f is concave up, shaped like a cup, and where $f''(x) < 0$, f is concave down, shaped like a cap. A **point of inflection** is a point where the concavity changes and the tangent line exists there. The second part of the definition matters: at $x = 0$, $f(x) = x^{1/3}$ changes concavity but has a vertical tangent, so it is not a point of inflection on the exam.

To find inflection points, locate where f'' is zero or undefined, keep only the points where f is continuous, and check that the sign of f'' actually changes. For $f(x) = x^4$, $f''(x) = 12x^2$ is zero at $x = 0$ but positive on both sides, so there is no inflection point.

Trap. $f''(c) = 0$ does not make c a point of inflection by itself. The sign of f'' must change across c . The quartic x^4 is the standard counterexample that catches students who skip the sign check.

5.7 Using the Second Derivative Test to Determine Extrema

The **Second Derivative Test** classifies a critical point c where $f'(c) = 0$. If $f''(c) > 0$, the graph is concave up there and f has a local minimum. If $f''(c) < 0$, the graph is concave down and f has a local maximum. If $f''(c) = 0$, the test is inconclusive and you must use the First Derivative Test instead. The test also cannot be used at a critical point where the derivative is undefined, since f'' cannot be evaluated there.

For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$ gives critical points $x = \pm 1$, and $f''(x) = 6x$. Since $f''(-1) = -6 < 0$, there is a local maximum at $x = -1$. Since $f''(1) = 6 > 0$, there is a local minimum at $x = 1$. The values are $f(-1) = 2$ and $f(1) = -2$.

Trap. When $f''(c) = 0$, the test says nothing. For $f(x) = x^4$, $f''(0) = 0$, yet $x = 0$ is a local minimum. For $f(x) = -x^4$, $f''(0) = 0$, yet $x = 0$ is a local maximum. For $f(x) = x^3$, $f''(0) = 0$ and there is no extremum at all. One test value, three different outcomes.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Local vs absolute extremum	Local compares against nearby values. Absolute compares against every value on the interval or domain. A local maximum can sit well below the absolute maximum.
First vs Second Derivative Test	The First Test reads the sign change of f' and works at any critical point. The Second Test reads f'' at a point where $f' = 0$, and goes silent when $f'' = 0$ or f' is undefined.
Critical point vs extremum	Every interior local extremum is a critical point, but the converse is false. A critical point with no sign change of f' is neither a max nor a min.
Concavity change vs inflection point	An inflection point needs a concavity change plus an existing tangent line. The cusp of $x^{1/3}$ changes concavity at 0 but has a vertical tangent, so it is excluded.
MVT vs EVT	The MVT promises a point where the derivative equals a secant slope, and needs differentiability on (a, b) . The EVT promises that absolute extrema exist, and needs only continuity on $[a, b]$.
Candidates Test vs First Derivative Test	The Candidates Test finds absolute extrema on a closed interval by comparing values. The First Derivative Test finds local extrema anywhere by reading sign changes.

5.10 Introduction to Optimization Problems

Optimization means finding the largest or smallest value a quantity can take. The method is to write the quantity as a function of a single variable, restrict to the relevant domain, and use derivatives to locate the extrema. The setup is where most points are won or lost: name the variables, write the constraint equation, substitute into the quantity, and only then differentiate.

A farmer has 100 feet of fence to enclose a rectangular pen against a barn, so only three sides need fencing. Let x be the length of the two sides perpendicular to the barn and y the side parallel to it. The constraint is $2x + y = 100$, so $y = 100 - 2x$. The area is $A = xy = x(100 - 2x) = 100x - 2x^2$ for $0 \leq x \leq 50$. Differentiating, $A' = 100 - 4x$, which is zero at $x = 25$. Since $A'' = -4 < 0$, this is a maximum, and the endpoints give $A = 0$, so $x = 25$ is the absolute maximum. The pen is 25 feet by 50 feet with area 1250 square feet.

Trap. Justify that the extremum is absolute, not just local. Checking the endpoints of the domain, or noting the sign of the second derivative together with a single critical point, is what turns a local result into the answer the question asks for. A critical point alone never proves a global maximum.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. The function $f(x) = x^2$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$. The Mean Value Theorem guarantees a number c in $(1, 3)$ such that $f'(c) =$
A. 1
B. 2
C. 4
D. No such number exists
2. Let $f(x) = x(x - 2)^2$. Which statement is true?
A. f has a local maximum at $x = 0$ and a local minimum at $x = 2$
B. f has a local minimum at $x = 0$ and no local extremum at $x = 2$
C. f has a local minimum at both $x = 0$ and $x = 2$
D. f has a local maximum at $x = 2$ and no local extremum at $x = 0$
3. For $f(x) = x^3 - 3x$, the Second Derivative Test shows that
A. f has a local maximum at $x = 1$ and a local minimum at $x = -1$
B. f has a local maximum at $x = -1$ and a local minimum at $x = 1$
C. f has a local minimum at both $x = -1$ and $x = 1$
D. the test is inconclusive at both $x = -1$ and $x = 1$
4. A farmer uses 100 feet of fence to enclose a rectangular pen against a barn, so only three sides are fenced. If x is the length in feet of each side perpendicular to the barn, the value of x that maximizes the enclosed area is
A. 20
B. 25
C. 50
D. 100

Answer Key

- 1. C.** The MVT says $f'(c)$ equals the average rate of change, $(f(3) - f(1)) / (3 - 1) = (9 - 1) / 2 = 4$. Since $f'(x) = 2x$, solving $2c = 4$ gives $c = 2$, which lies in $(1, 3)$ as required. A divides incorrectly. B reports the c value, not the required $f'(c)$ value. D ignores that the hypotheses are satisfied, so the conclusion holds.
- 2. B.** The critical points are $x = 0$ and $x = 2$. For $x < 0$, $f' < 0$; between 0 and 2, $f' > 0$; past 2, $f' > 0$. At $x = 0$ the sign changes from negative to positive, giving a local minimum by the First Derivative Test. At $x = 2$ the sign stays positive on both sides, so there is no local extremum. A swaps the classification and invents a minimum at $x = 2$. C ignores the missing sign change at $x = 2$. D misreads both sign changes.
- 3. B.** $f'(x) = 3x^2 - 3$, so the critical points are $x = \pm 1$. $f''(x) = 6x$. At $x = -1$, $f''(-1) = -6 < 0$, so the graph is concave down and f has a local maximum. At $x = 1$, $f''(1) = 6 > 0$, so the graph is concave up and f has a local minimum. A reverses the two. C confuses the sign of the second derivative. D would require $f'' = 0$ at the critical points, but it is ± 6 .
- 4. B.** With the constraint $2x + y = 100$, the area is $A = xy = x(100 - 2x) = 100x - 2x^2$ on $0 \leq x \leq 50$. Setting $A' = 100 - 4x = 0$ gives $x = 25$. The endpoints give $A = 0$, and $A'' = -4 < 0$, so $x = 25$ is the absolute maximum. A halves the correct answer. C uses the full 100 feet for the parallel side alone, ignoring the constraint. D uses all the fence for one perpendicular side, leaving no area.

One-Page Recall Check

- State the Mean Value Theorem with both hypotheses and the exact conclusion.
- State the Extreme Value Theorem and name one way each hypothesis can fail.
- Define local extremum, absolute extremum, and critical point in your own words.
- Explain why a critical point is not automatically an extremum, with an example.
- Describe the sign-chart procedure for finding increasing and decreasing intervals.
- State the First Derivative Test and explain when it is preferable to the Second Derivative Test.
- Walk through the Candidates Test for a continuous function on $[a, b]$, step by step.
- Define concavity and point of inflection, and explain why the tangent line must exist.
- State the Second Derivative Test including the inconclusive case, with one example of each outcome.
- Outline the optimization setup: variables, constraint, single-variable function, derivative, justification.
- Redo the fence problem from scratch, showing the constraint substitution and the endpoint check.
- Find the local extrema of $f(x) = x^3 - 3x$ using both derivative tests.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Analytical Applications of Differentiation deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Mean Value Theorem (MVT), Extreme Value Theorem (EVT), Local (relative) extremum, Absolute (global) extremum, Critical point, Increasing and decreasing intervals, First Derivative Test, Candidates Test, Concavity, Point of inflection, Second Derivative Test, Optimization.

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