

Unit 4: Contextual Applications of Differentiation

Unit 4 turns the derivative into a tool for answering questions about the world. It covers motion along a line, rates that change together in time, tangent-line approximations, and L'Hôpital's Rule for indeterminate limits. The common thread is that every topic starts with a relationship between quantities and asks what the derivative tells you about it.

AP Calculus BC

Contextual Applications of Differentiation

About 12 minutes to read

How to use this guide

Read it in order the first time because the problem-solving habits build on each other. Motion problems train you to read the sign of derivatives, related-rates problems train you to differentiate before substituting, linearization trains you to work with tangent lines as estimates, and L'Hôpital's Rule gives you a new way to handle the $0/0$ and ∞/∞ forms from Unit 1. Exam questions here are mostly word problems, so practice translating the sentences into equations.

After the first read, use the trap boxes and the confusions table to review the distinctions the exam tests most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Contextual Applications of Differentiation is about 10 to 15 percent of the AP Calculus BC exam. The topics show up on both the multiple-choice and free-response sections, and related rates is one of the most reliable free-response formats. Straight-line motion and L'Hôpital's Rule also feed directly into later units.

4.2 Straight-Line Motion: Position, Velocity, and Acceleration

The **position function** $s(t)$ gives an object's location at time t along a line. The **velocity** is its derivative, $v(t) = s'(t)$, and it is the instantaneous rate of change of position. The sign of the velocity tells you the direction of motion. Positive velocity means moving in the positive direction, negative velocity means moving in the negative direction, and zero velocity means the object is instantaneously at rest.

The **acceleration** is the derivative of velocity, $a(t) = v'(t) = s''(t)$. It measures how velocity itself is changing. The **speed** is the magnitude of velocity, $|v(t)|$, and it is always nonnegative. Speed answers "how fast" while velocity answers "how fast and which way."

Worked example. Let $s(t) = t^3 - 6t^2 + 9t + 1$ for $t \geq 0$. Differentiating gives $v(t) = 3t^2 - 12t + 9$, which factors as $3(t - 1)(t - 3)$. The velocity is zero at $t = 1$ and $t = 3$, positive on $(0, 1)$ and $(3, \infty)$, and negative on $(1, 3)$. So the particle moves right, then turns at $t = 1$, moves left until $t = 3$, turns again, and moves right afterward. Differentiating again gives $a(t) = 6t - 12$, so the acceleration is zero at $t = 2$, negative before, and positive after.

Trap. Velocity and speed are not the same quantity. If $v(2) = -3$ m/s, the velocity is -3 m/s but the speed is 3 m/s. Questions that ask "how fast" want the absolute value. Questions that ask about direction want the sign.

Trap. The speed is decreasing exactly when velocity and acceleration have opposite signs. In the example above, on $(1, 2)$ the velocity is negative and the acceleration is negative, so the speed is increasing. On $(2, 3)$ the velocity is negative but the acceleration is positive, so the speed is decreasing. Checking the sign of $a(t)$ alone does not tell you about speed.

4.4 Related Rates

Related rates problems ask how fast one quantity is changing when you know how fast a related quantity is changing. The quantities are linked by a geometric or physical equation, and because everything changes with time, you differentiate implicitly with respect to t and then substitute the known values.

The order of operations matters. First write the equation relating the quantities. Then differentiate both sides with respect to t , using the chain rule so that every variable gets its own rate. Only after differentiating do you substitute the known instantaneous values. Substituting before differentiating freezes a variable that is actually changing, and the rate you need disappears.

Worked example. Air is pumped into a spherical balloon so its volume increases at $12\pi \text{ cm}^3/\text{s}$. How fast is the radius increasing when $r = 2 \text{ cm}$? The volume is $V = (4/3)\pi r^3$. Differentiating with respect to t gives $dV/dt = 4\pi r^2 \cdot dr/dt$. Substituting $dV/dt = 12\pi$ and $r = 2$ gives $12\pi = 4\pi(2)^2 \cdot dr/dt$, so $12\pi = 16\pi \cdot dr/dt$ and $dr/dt = 12\pi / 16\pi = 3/4 \text{ cm/s}$.

Trap. Differentiate first, substitute second. If you plug $r = 2$ into $V = (4/3)\pi r^3$ before differentiating, you get the constant $V = (32/3)\pi$, whose derivative is 0, and the problem is destroyed. The general equation keeps every rate visible.

4.5 Related Rates: the Ladder Problem

A 10-foot ladder slides down a wall. The bottom moves away from the wall at 2 ft/s . How fast is the top sliding down when the bottom is 6 ft from the wall? Let x be the distance from the wall and y the height. Then $x^2 + y^2 = 100$. Differentiating gives $2x \cdot dx/dt + 2y \cdot dy/dt = 0$. When $x = 6$, $y = \sqrt{100 - 36} = 8$. Substituting gives $2(6)(2) + 2(8) \cdot dy/dt = 0$, so $24 + 16 \cdot dy/dt = 0$ and $dy/dt = -3/2 \text{ ft/s}$. The negative sign means the height is decreasing, so the top slides down at 1.5 ft/s .

Trap. A negative rate is information, not an error. Here $dy/dt = -3/2 \text{ ft/s}$ tells you the top of the ladder is moving downward at 1.5 ft/s . Read the sign as the direction of change, and report the speed as a positive number with a direction word.

4.6 Local Linearity and Linearization

Local linearity is the observation that a differentiable function looks like a straight line when you zoom in close enough at a point. The **linearization**, or tangent line approximation, turns that observation into a formula. Near $x = a$, the function values are approximately $f(x) \approx f(a) + f'(a)(x - a)$, which is the line through $(a, f(a))$ with slope $f'(a)$.

This is useful because the tangent line is easy to evaluate even when the function is not. The approximation is good near a and gets worse as x moves away, so choose a point where you can compute $f(a)$ and $f'(a)$ exactly and where x is close to a .

Worked example. Approximate $\sqrt{4.1}$ using linearization at $a = 4$. Let $f(x) = \sqrt{x}$. Then $f(4) = 2$ and $f'(x) = 1/(2\sqrt{x})$, so $f'(4) = 1/(2 \cdot 2) = 1/4$. The linearization is $L(x) = 2 + (1/4)(x - 4)$. Evaluating at $x = 4.1$ gives $L(4.1) = 2 + (1/4)(0.1) = 2.025$. The true value is $\sqrt{4.1} \approx 2.0248$, so the estimate is off by about 0.0002, exactly what you expect this close to a .

Trap. The linearization needs $f'(a)$, not $f'(x)$. A common slip is to write $L(x) = f(a) + f'(x)(x - a)$ with the derivative still varying. The whole point is that $f'(a)$ is a single number, the slope of one specific tangent line.

4.7 L'Hôpital's Rule and Indeterminate Forms

An **indeterminate form** is a limit form like $0/0$ or ∞/∞ whose value cannot be read from the form alone. The form $0/0$ might equal anything, which is why it needs further analysis.

L'Hôpital's Rule handles these forms. If $\lim f(x)/g(x)$ is $0/0$ or ∞/∞ as x approaches c (or $\pm\infty$), and f and g are differentiable near c , then the limit equals $\lim f'(x)/g'(x)$, provided that limit exists. You differentiate the top and bottom separately, not the quotient. If the result is still indeterminate, apply the rule again.

Worked example. Find $\lim_{x \rightarrow 0} \sin x / x$. Direct substitution gives $0/0$, so the form is indeterminate.

Differentiating top and bottom separately gives $\lim_{x \rightarrow 0} \cos x / 1 = 1/1 = 1$. This is the fundamental trigonometric limit that appears throughout the course.

Worked example. Find $\lim_{x \rightarrow 0} (e^x - 1 - x) / x^2$. Direct substitution gives $(1 - 1 - 0)/0 = 0/0$. Applying L'Hôpital's Rule once gives $\lim_{x \rightarrow 0} (e^x - 1) / (2x)$, which is still $0/0$. Applying it a second time gives $\lim_{x \rightarrow 0} e^x / 2 = 1/2$. Two applications were needed because the first one stayed indeterminate.

Trap. L'Hôpital's Rule differentiates the numerator and denominator separately. It is not the quotient rule. Writing $d/dx [f(x)/g(x)]$ as $(f'g - fg')/g^2$ is a different operation entirely and will not resolve the limit.

Trap. Check the form before applying the rule. L'Hôpital's Rule only applies to $0/0$ or ∞/∞ . If direct substitution gives a number, or a nonzero number over 0, the rule does not apply, and using it anyway produces a wrong answer.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Velocity vs speed	Velocity is signed and gives direction. Speed is $ v(t) $ and is never negative. "How fast" usually means speed.
Speed increasing vs velocity increasing	Speed increases when v and a share a sign. Velocity increases when a is positive, regardless of the sign of v .
Differentiate first vs substitute first	In related rates, write the general equation, differentiate with respect to t , then substitute. Substituting first turns variables into constants.
Linearization $f'(a)$ vs $f'(x)$	The approximation uses the slope at the single point a . A varying derivative defeats the purpose of the tangent line.
L'Hôpital vs quotient rule	L'Hôpital differentiates top and bottom separately. The quotient rule differentiates the fraction as a whole. They solve different problems.
$0/0$ vs a number over 0	$0/0$ is indeterminate and invites L'Hôpital. A nonzero number over 0 is an infinite limit, and L'Hôpital does not apply.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t + 1$ for $t \geq 0$. On which interval is the particle moving to the left?
 - A. $0 < t < 1$
 - B. $1 < t < 3$
 - C. $t > 3$
 - D. The particle never moves left
2. Air is pumped into a spherical balloon so that its volume increases at a rate of $12\pi \text{ cm}^3/\text{s}$. At what rate is the radius increasing when the radius is 2 cm?
 - A. $3/4 \text{ cm/s}$
 - B. $3\pi \text{ cm/s}$
 - C. $1/4 \text{ cm/s}$
 - D. 12 cm/s
3. Let $f(x) = \sqrt{x}$. The linearization of f at $x = 4$ is used to approximate $\sqrt{4.1}$. What is the approximate value?
 - A. 2.025
 - B. 2.05
 - C. 2.0025
 - D. 2.25
4. $\lim_{x \rightarrow 0} (e^x - 1 - x) / x^2 =$
 - A. 0
 - B. $1/2$
 - C. 1
 - D. ∞

Answer Key

- 1. B.** Velocity is $v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$. Moving left means $v(t) < 0$, which happens when exactly one factor is negative, on $(1, 3)$. A gives the interval where the particle moves right ($v > 0$). C is also a rightward interval. D ignores that the velocity changes sign twice.
- 2. A.** From $V = (4/3)\pi r^3$, implicit differentiation gives $dV/dt = 4\pi r^2 \cdot dr/dt$. Substituting $dV/dt = 12\pi$ and $r = 2$ gives $12\pi = 16\pi \cdot dr/dt$, so $dr/dt = 3/4$ cm/s. B comes from dividing 12π by 4 alone, forgetting both the π in $4\pi r^2$ and the $r^2 = 4$ substitution, which gives $12\pi/4 = 3\pi$. C might come from using the diameter instead of the radius. D just repeats the given volume rate without differentiating at all.
- 3. A.** With $f(4) = 2$ and $f'(4) = 1/(2\sqrt{4}) = 1/4$, the linearization is $L(x) = 2 + (1/4)(x - 4)$. Then $L(4.1) = 2 + (1/4)(0.1) = 2.025$. B uses $f'(4) = 1/2$, forgetting the 2 in the derivative of $\sqrt{x}$. C is a decimal slip. D is far too large and would mean the approximation ignored how close 4.1 is to 4.
- 4. B.** Direct substitution gives $0/0$. One application of L'Hôpital's Rule gives $(e^x - 1)/(2x)$, still $0/0$. A second application gives $e^x/2$, which approaches $1/2$. A stops after one application and treats the still-indeterminate form as an answer. C might come from applying L'Hôpital once and then treating $(e^x - 1)/(2x)$ at $x = 0$ as though $0/0$ equals 1. D comes from differentiating the numerator twice and the denominator once, giving $e^x/(2x)$, which blows up.

One-Page Recall Check

- State the position, velocity, and acceleration functions and how each is obtained from the previous one.
- Explain how the sign of velocity determines direction of motion, and when the object is at rest.
- Define speed and explain why it differs from velocity.
- State the rule for when speed is increasing versus decreasing in terms of $v(t)$ and $a(t)$.
- Describe the three steps of a related-rates problem in order, and explain why substitution comes last.
- Work the balloon problem from scratch: $V = (4/3)\pi r^3$, $dV/dt = 12\pi$, find dr/dt at $r = 2$.
- Work the ladder problem from scratch and explain what the negative sign in the answer means.
- State the local linearity idea and write the linearization formula $L(x) = f(a) + f'(a)(x - a)$.
- Use linearization at $x = 4$ to approximate $\sqrt{4.1}$, showing every step.
- Define an indeterminate form and list the two forms L'Hôpital's Rule handles.
- State L'Hôpital's Rule precisely, including the requirement that the new limit exists.
- Evaluate $\lim_{x \rightarrow 0} \sin x / x$ using L'Hôpital's Rule.
- Evaluate $\lim_{x \rightarrow 0} (e^x - 1 - x) / x^2$, explaining why two applications are needed.
- Explain the difference between L'Hôpital's Rule and the quotient rule.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Contextual Applications of Differentiation deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Position function, Velocity, Acceleration, Speed, Related rates, Local linearity, Linearization (tangent line approximation), L'Hôpital's Rule, Indeterminate form.

About this guide. Written for Rycal and aligned to the College Board AP Calculus BC course framework, Unit 4. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.