

# Unit 3: Differentiation: Composite, Implicit, and Inverse Functions

Unit 3 extends differentiation to functions built from other functions. It covers the chain rule for composites, implicit differentiation for equations that mix  $x$  and  $y$ , the derivative of an inverse function, the derivatives of the inverse trigonometric functions, and higher-order derivatives.

AP Calculus BC

Differentiation: Composite, Implicit, and Inverse Functions

About 10 minutes to read

## How to use this guide

Read it in order the first time. The chain rule comes first because implicit differentiation is the chain rule applied to  $y$ -terms, and the inverse-function rule is best understood as the chain rule applied to  $f(f^{-1}(x)) = x$ . Work each example on paper before reading the solution. The exam tests whether you apply the inner derivative every time, not whether you can state the rule.

**What this unit is worth.** This unit is about 4 to 7 percent of the AP Calculus BC exam. Like Unit 2, it is a tools unit, and the chain rule in particular shows up inside nearly every later topic, including related rates, differential equations, and series.

## 3.1 The Chain Rule

A **composite function** is one function applied to the output of another, written  $f(g(x))$  or  $(f \circ g)(x)$ . For example,  $(3x^2 + 1)^4$  is the fourth-power function applied to the polynomial  $3x^2 + 1$ , and  $\sin(2x)$  is the sine function applied to  $2x$ .

The **chain rule** says that for  $f(g(x))$ , the derivative is  $f'(g(x)) \cdot g'(x)$ . In words, take the derivative of the outside, then multiply by the derivative of the inside. The inner derivative is the part students skip, and the exam counts on that.

Example: differentiate  $(3x^2 + 1)^4$ . The outside is  $u^4$ , whose derivative is  $4u^3$ . The inside is  $u = 3x^2 + 1$ , whose derivative is  $6x$ . Multiply:  $4(3x^2 + 1)^3 \cdot 6x = 24x(3x^2 + 1)^3$ .

Example: differentiate  $\sin(2x)$ . The outside is  $\sin u$ , whose derivative is  $\cos u$ . The inside is  $u = 2x$ , whose derivative is  $2$ . So the result is  $\cos(2x) \cdot 2 = 2\cos(2x)$ .

**Trap.** The answer  $4(3x^2 + 1)^3$  is wrong for the first example above. It is the derivative of the outside only. The chain rule is not finished until you multiply by the derivative of the inside. Every distractor on chain-rule questions is built from answers that skip that step.

## 3.2 Implicit Differentiation

**Implicit differentiation** handles equations where  $y$  is not solved for, such as  $x^2 + y^2 = 25$ . Treat  $y$  as a function of  $x$ . When you differentiate a  $y$ -term, the chain rule produces a  $dy/dx$  factor. Then solve the resulting equation for  $dy/dx$ .

Example:  $x^2 + y^2 = 25$ . Differentiating term by term gives  $2x + 2y \cdot dy/dx = 0$ , because the derivative of  $y^2$  with respect to  $x$  is  $2y$  times  $dy/dx$  by the chain rule. Solving:  $dy/dx = -2x / 2y = -x/y$ .

Example:  $xy + y^2 = 1$ . The product  $xy$  needs the product rule: its derivative is  $y + x \cdot dy/dx$ . The  $y^2$  term gives  $2y \cdot dy/dx$ . So  $y + x \cdot dy/dx + 2y \cdot dy/dx = 0$ . Collect the  $dy/dx$  terms:  $dy/dx (x + 2y) = -y$ , and  $dy/dx = -y / (x + 2y)$ .

**Trap.** Differentiating  $y^2$  as  $2y$  is the single most common error here. The derivative is  $2y \cdot dy/dx$ . Any  $y$ -term always picks up a  $dy/dx$  factor because  $y$  is a function of  $x$ . If your answer has no  $dy/dx$  in it, you missed one.

### 3.3 Differentiating Inverse Functions

The **derivative of an inverse function** follows from the definition. If  $f(a) = b$  and  $f$  is invertible, then  $(f^{-1})'(b) = 1 / f'(a)$ , provided  $f'(a) \neq 0$ . The slope of the inverse at the point  $b$  is the reciprocal of the slope of the original function at the point  $a$ .

The way to use it: find the  $x$ -value  $a$  that maps to  $b$ , evaluate  $f'$  there, and take the reciprocal. The evaluation happens at  $a$ , the input to  $f$ , not at  $b$ .

Example:  $f(x) = x^3 + 2$ . Find  $(f^{-1})'(3)$ . First find  $a$  with  $f(a) = 3$ . Since  $1^3 + 2 = 3$ ,  $a = 1$ . Then  $f'(x) = 3x^2$ , so  $f'(1) = 3$ . Therefore  $(f^{-1})'(3) = 1/3$ .

**Trap.** The formula needs  $f'(a)$ , evaluated at the  $x$ -value of the original function. Computing  $1 / f'(3)$  instead of  $1 / f'(1)$  in the example above gives  $1/27$ , which is wrong. The first step of every inverse-derivative problem is finding  $a$ , the value with  $f(a) = b$ .

### 3.4 Differentiating Inverse Trigonometric Functions

The three inverse trigonometric derivatives come from implicit differentiation of the inverse relationship, and they are worth memorizing because the exam treats them as known rules.

| FUNCTION    | DERIVATIVE            |
|-------------|-----------------------|
| $\arcsin x$ | $1 / \sqrt{1 - x^2}$  |
| $\arccos x$ | $-1 / \sqrt{1 - x^2}$ |
| $\arctan x$ | $1 / (1 + x^2)$       |

When the argument is itself a function, apply the chain rule. Example: differentiate  $\arcsin(2x)$ . The derivative of  $\arcsin u$  is  $1 / \sqrt{1 - u^2}$ , and the inner derivative is  $2$ , so the result is  $2 / \sqrt{1 - 4x^2}$ .

**Trap.** The  $\arcsin$  and  $\arccos$  derivatives differ only by a sign. The  $\arcsin$  rule has no negative; the  $\arccos$  rule does. Students also mix up the square roots, writing  $1 / \sqrt{1 + x^2}$  for  $\arcsin$ . The plus belongs to  $\arctan$ , which has no square root at all.

## 3.6 Calculating Higher-Order Derivatives

Note. Topic 3.5 (selecting procedures for calculating derivatives) has no separate section here. Choosing the right rule is practiced in every example and question above.

The **second derivative** is the derivative of the derivative:  $f''(x)$ ,  $y''$ , or  $d^2y/dx^2$ . It measures how the rate of change itself changes. For  $f(x) = x^4$ ,  $f'(x) = 4x^3$  and  $f''(x) = 12x^2$ .

**Higher-order derivatives** continue the pattern:  $f'''(x)$ ,  $f^{(4)}(x)$ , and in general  $f^{(n)}(x)$  for the  $n$ th derivative. Just differentiate again, applying the same rules each time. There is no shortcut that skips a step.

Example:  $f(x) = x^4 - 2x^2$ . Then  $f'(x) = 4x^3 - 4x$  and  $f''(x) = 12x^2 - 4$ . Each pass uses the power rule on the result of the previous pass.

**Trap.** The second derivative is not the derivative of  $f'(x)$  with a fresh power rule shortcut. It is  $f'$  differentiated normally, which for  $f(x) = x^4 - 2x^2$  gives  $12x^2 - 4$ , not  $4x^3 - 4x$  written again. Always do both passes on paper.

## Confusions That Cost Points

| PAIR                                                 | HOW TO KEEP THEM STRAIGHT                                                                                                  |
|------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|
| Chain rule with and without the inner derivative     | $d/dx[(3x^2 + 1)^4] = 24x(3x^2 + 1)^3$ . Without the $6x$ from the inside, the answer is incomplete.                       |
| $2y \cdot dy/dx$ vs $2y$ in implicit differentiation | Any $y$ -term picks up $dy/dx$ because $y$ is a function of $x$ . An answer with no $dy/dx$ missed a chain-rule step.      |
| $1 / f'(a)$ vs $1 / f'(b)$ for inverses              | The inverse rule evaluates $f'$ at $a$ , the input with $f(a) = b$ . Find $a$ first, then differentiate, then reciprocate. |
| $\arcsin$ vs $\arccos$ vs $\arctan$ derivatives      | $\arcsin$ : $1/\sqrt{1-x^2}$ . $\arccos$ : same with a negative. $\arctan$ : $1/(1+x^2)$ , no root.                        |
| $f''(x)$ vs $f'(x)$                                  | The second derivative requires a full second pass of differentiation. Write both passes out.                               |

## Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1.  $\frac{d}{dx}[(2x^3 - 1)^4] =$

- A.  $4(2x^3 - 1)^3$
- B.  $24x^2(2x^3 - 1)^3$
- C.  $24x^2(2x^3 - 1)^2$
- D.  $4(6x^2)^3$

2. For  $x^2 + 3xy = 1$ ,  $dy/dx =$

- A.  $-(2x + 3y) / (3x)$
- B.  $-(2x + 3y) / 3$
- C.  $2x + 3y$
- D.  $(-2x - 3y) / (3x + 3y)$

3. Let  $f(x) = x^3 + 2x$ . Then  $(f^{-1})'(0) =$

- A. 2
- B.  $1/2$
- C. 0
- D.  $-1/2$

4. If  $f(x) = x^4 - 2x^2$ , then  $f''(x) =$

- A.  $4x^3 - 4x$
- B.  $12x^2 - 4$
- C.  $12x^3 - 4$
- D.  $24x$

## Answer Key

- 1. B.** The outside derivative is  $4(2x^3 - 1)^3$  and the inside derivative is  $6x^2$ . Multiply:  $4(2x^3 - 1)^3 \cdot 6x^2 = 24x^2(2x^3 - 1)^3$ . A is the outside derivative alone, the classic skipped inner derivative. C keeps the wrong power, which happens when the outside is differentiated carelessly. D differentiates the inside first and then cubes it, which is not how the chain rule works.
- 2. A.** Differentiating:  $2x + 3y + 3x \cdot dy/dx = 0$ , since  $d/dx[3xy] = 3y + 3x \cdot dy/dx$  by the product rule. Collect:  $3x \cdot dy/dx = -2x - 3y$ , so  $dy/dx = -(2x + 3y) / (3x)$ . B drops the  $x$  from the denominator, which comes from collecting the  $dy/dx$  terms incorrectly. C never solves for  $dy/dx$  at all. D adds a  $3y$  to the denominator, which is not a term that multiplies  $dy/dx$ .
- 3. B.** Find  $a$  with  $f(a) = 0$ . Since  $f(0) = 0$ ,  $a = 0$ . Then  $f'(x) = 3x^2 + 2$ , so  $f'(0) = 2$ , and  $(f^{-1})'(0) = 1/2$ . A reports  $f'(0)$  without taking the reciprocal. C confuses the input  $b = 0$  with the answer. D adds a negative sign that appears nowhere in the reciprocal rule.
- 4. B.** First pass:  $f'(x) = 4x^3 - 4x$ . Second pass:  $f''(x) = 12x^2 - 4$ . A stops after one derivative. C differentiates the power wrong on the second pass ( $3 \cdot 4x^3$  would need an  $x^4$  term, which is not there). D is the third derivative, one pass too many.

# One-Page Recall Check

- Write the chain rule for  $f(g(x))$  in words and in symbols.
- Differentiate  $(5x^2 - 3)^6$  from scratch, showing the inner derivative step.
- Explain why differentiating  $y^2$  with respect to  $x$  gives  $2y \cdot dy/dx$  and not  $2y$ .
- Find  $dy/dx$  for  $x^2 + y^2 = 16$ , showing the solve step.
- State the inverse-function derivative rule and explain why  $f'$  is evaluated at  $a$ , not  $b$ .
- For  $f(x) = x^3 + x$  with  $f(0) = 0$ , compute  $(f^{-1})'(0)$  by hand.
- Write the derivatives of  $\arcsin x$ ,  $\arccos x$ , and  $\arctan x$  from memory.
- Differentiate  $\arctan(3x)$ , showing the chain-rule factor.
- Explain how the second derivative differs from the first, and compute  $f''(x)$  for  $f(x) = x^5$ .
- Name two errors from the trap boxes that you have made before, and say how you will catch them next time.

**Where to go next.** Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the AP Calculus BC deck. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

## Key terms for this unit

Chain Rule, Composite function, Implicit differentiation, Derivative of an inverse function, Derivative of  $\arcsin x$ , Derivative of  $\arctan x$ , Derivative of  $\arccos x$ , Second derivative, Higher-order derivatives.

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