

Unit 2: Differentiation: Definition and Fundamental Properties

Unit 2 turns the limit into the derivative. It covers average and instantaneous rates of change, the definition of the derivative, the notation you must use, differentiability and its connection to continuity, and the basic differentiation rules: power, constant, sum, difference, constant multiple, product, quotient, and the derivatives of the six trig functions, e^x , and $\ln x$.

AP Calculus BC

Differentiation: Definition and Fundamental Properties

About 12 minutes to read

How to use this guide

Read it in order the first time because the ideas build. Rates of change motivate the derivative, the derivative needs notation, notation needs the rules, and the rules need practice. Work the examples with a pencil. The practice questions at the end target the exact mistakes students make most often on this unit.

After the first read, use the trap boxes and the rules table to drill the distinctions. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Differentiation fundamentals are about 10 to 12 percent of the AP Calculus BC exam, but the real weight is larger. Every later unit uses these rules. If the power rule or the product rule feels shaky now, the chain rule, related rates, and everything after will feel shaky too.

2.1 Defining Average and Instantaneous Rates of Change at a Point

The **average rate of change** of f over the interval $[a, b]$ is $[f(b) - f(a)] / (b - a)$. On a graph it is the slope of the **secant line** through the two points $(a, f(a))$ and $(b, f(b))$. For $f(x) = x^2$ over $[1, 3]$, the average rate is $(9 - 1) / (3 - 1) = 8 / 2 = 4$.

The **instantaneous rate of change** at a point c is what you get when the interval shrinks down to c . On a graph it is the slope of the **tangent line** through $(c, f(c))$. The tangent line touches the curve at that single point and follows its direction there. Its slope is $f'(c)$, the derivative at c .

Trap. The secant slope and the tangent slope answer different questions. The secant gives the average over an interval. The tangent gives the rate at one point. For $f(x) = x^2$, the average over $[1, 3]$ is 4, but the instantaneous rate at $x = 1$ is 2. Do not report one when the question asks for the other.

2.2 Defining the Derivative of a Function and Using Derivative Notation

The **derivative of a function** at x is defined by the limit $f'(x) = \lim_{h \rightarrow 0} [f(x + h) - f(x)] / h$, when that limit exists. It is the instantaneous rate of change of f at x . You should be able to write this definition from memory. Justification questions on the exam expect it.

Derivative notation comes in several forms that all mean the same thing: $f'(x)$, y' , dy/dx , and $d/dx[f(x)]$. The dy/dx form is **Leibniz notation** and $f'(x)$ is **Lagrange (prime) notation**. Read dy/dx as "the derivative of y with respect to x ." On free-response questions, write the notation every time you differentiate. Bare algebra with no notation loses credit.

Trap. The definition is a limit, not a fraction you can cancel. Writing $[f(x + h) - f(x)] / h$ and then setting $h = 0$ without taking the limit is the most common way to lose the point on a definition question. The limit is doing the work.

2.3 Estimating Derivatives of a Function at a Point

The **difference quotient** $[f(x + h) - f(x)] / h$ (or $[f(a + h) - f(a)] / h$ at a point a) is the slope of the secant line. Its limit as $h \rightarrow 0$ is the derivative. Before the limit is taken, the difference quotient gives an estimate of the derivative at the point. Smaller values of h give better estimates.

You can also estimate a derivative from a table or a graph. From a table, compute $[f(b) - f(a)] / (b - a)$ using the two entries closest to the point. From a graph, sketch the tangent line at the point and estimate its slope with rise over run between two readable points on your sketch.

Trap. A table estimate is only as good as the spacing of the data. If the table jumps by 0.5, your estimate of $f'(2)$ is really an average over a wide interval. Say so if the question asks about accuracy. Do not present a table estimate as exact.

2.4 Connecting Differentiability and Continuity

A function is **differentiable** at c when $f'(c)$ exists, which means the limit defining the derivative exists there. Geometrically, the graph has a well-defined tangent line at that point, not a corner or a break.

Differentiability implies continuity. If f is differentiable at c , then f is continuous at c . The converse is false. A function can be continuous but not differentiable at a point. The three ways this happens are corners (like $|x|$ at $x = 0$, where the left and right slopes disagree), cusps (sharp points where the slopes blow up in opposite directions), and vertical tangents (where the slope is undefined because the tangent line is vertical).

Trap. Students reverse the implication. "Continuous" does not mean "differentiable." The absolute value function is continuous everywhere and fails to be differentiable at $x = 0$. When a question asks whether differentiability follows from continuity, the answer is no, and $|x|$ is the counterexample to cite.

2.5 Applying the Power Rule

The **Power Rule** says that if $f(x) = x^n$, then $f'(x) = n \cdot x^{n-1}$. Bring the exponent down in front and subtract one from the exponent. For $f(x) = x^5$, $f'(x) = 5x^4$. For $f(x) = x^3$, $f'(x) = 3x^2$.

The rule works for any real exponent n , including fractions and negatives. For $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = (1/2)x^{-1/2} = 1 / (2\sqrt{x})$. For $f(x) = 1/x = x^{-1}$, $f'(x) = -x^{-2} = -1/x^2$. Rewrite roots and fractions as powers before differentiating.

Trap. The power rule applies to x raised to a constant power. It does not apply to a constant raised to x . The derivative of 2^x is not $x \cdot 2^{x-1}$. Variable bases and variable exponents are different situations with different rules.

2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple

Four rules handle the basic combinations. The **Constant Rule** says the derivative of any fixed value is zero: $d/dx[c] = 0$. The **Sum Rule** says $d/dx[f + g] = f' + g'$. The **Difference Rule** says $d/dx[f - g] = f' - g'$. The **Constant Multiple Rule** says fixed values factor out: $d/dx[c \cdot f(x)] = c \cdot f'(x)$.

Used together they differentiate any polynomial term by term. For $f(x) = 5x^3 - 2x + 7$: the constant multiple rule gives $d/dx[5x^3] = 5 \cdot 3x^2 = 15x^2$, the sum and difference rules split the terms, and the constant rule kills the 7. The result is $f'(x) = 15x^2 - 2$.

Trap. The derivative of a constant is zero, but the constant multiple rule keeps the constant. In $5x^3$, the 5 stays and multiplies the derivative of x^3 . In the lone 7, there is no x attached, so the whole term differentiates to zero. Keep track of which constants are attached to x and which stand alone.

2.7 Derivatives of $\cos x$, $\sin x$, e^x , and $\ln x$

Four functions have derivatives worth memorizing cold. The **derivative of $\sin x$** is $\cos x$. The **derivative of $\cos x$** is $-\sin x$. Note the minus sign. It is the single most dropped sign in the unit.

The **derivative of e^x** is e^x . The exponential function is its own derivative, which is why it appears everywhere in growth models. The **derivative of $\ln x$** is $1/x$, for $x > 0$. The natural log is only defined for positive x , so its derivative carries that domain restriction.

FUNCTION	DERIVATIVE
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$1/x$

Trap. The minus sign on the derivative of $\cos x$ is easy to drop under time pressure, and it flips every answer that follows. When you see $\cos x$ in a product or quotient rule, write the $-\sin x$ first, then continue. Build the habit of pausing on that sign.

2.8 The Product Rule

The **Product Rule** says $d/dx[f \cdot g] = f' \cdot g + f \cdot g'$. Differentiate the first, leave the second. Then leave the first, differentiate the second. Add the two results. The order of the two terms does not matter, but both must appear.

For $f(x) = x^2 \cdot \sin x$: $f' = 2x$ and $g' = \cos x$, so $d/dx[x^2 \sin x] = 2x \cdot \sin x + x^2 \cdot \cos x$. Neither factor is a constant, so the constant multiple rule does not apply. Both factors get differentiated, each in its own term.

Trap. The derivative of a product is not the product of the derivatives. Writing $(x^2)' \cdot (\sin x)' = 2x \cos x$ misses half the answer. If both factors contain x , you need the product rule with both terms.

2.9 The Quotient Rule

The **Quotient Rule** says $d/dx[f / g] = (f' \cdot g - f \cdot g') / g^2$. Low d-high minus high d-low, over low squared. The minus sign in the numerator is the part students get wrong most often.

For $h(x) = (x^2 + 1) / (x - 3)$: the numerator differentiates to $2x$ and the denominator to 1 , so $h'(x) = [2x(x - 3) - (x^2 + 1)(1)] / (x - 3)^2 = (2x^2 - 6x - x^2 - 1) / (x - 3)^2 = (x^2 - 6x - 1) / (x - 3)^2$.

Trap. Two details decide quotient rule questions: the minus sign in the numerator and the squared denominator. Writing $(f'g + fg') / g$ is wrong twice over. Say the rule out loud as you write it: derivative of top times bottom, minus top times derivative of bottom, all over bottom squared.

2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

The remaining trig derivatives come from the quotient rule, but memorize the results. The **derivative of $\tan x$** is $\sec^2 x$. The **derivative of $\cot x$** is $-\csc^2 x$. The **derivative of $\sec x$** is $\sec x \cdot \tan x$. The **derivative of $\csc x$** is $-\csc x \cdot \cot x$.

Notice the pattern. The derivatives of $\tan$ and $\sec$ are positive. The derivatives of $\cot$ and $\csc$ carry a minus sign, the same way $\cos x$ does. The co-functions ($\cos$, $\cot$, $\csc$) are the ones with the negative sign.

FUNCTION	DERIVATIVE
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\csc x$	$-\csc x \cdot \cot x$

Trap. The $\sec$ and $\csc$ derivatives are the ones students mix up because each contains the other function. The derivative of $\sec x$ is $\sec x$ times $\tan x$. The derivative of $\csc x$ is negative $\csc x$ times $\cot x$. If you remember that the co-functions take the minus sign, you only have to memorize two of the four.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Average vs instantaneous rate	Average is the secant slope over an interval. Instantaneous is the tangent slope at a point, which is the derivative.
Differentiable vs continuous	Differentiable implies continuous. Continuous does not imply differentiable. Corners, cusps, and vertical tangents break differentiability only.
Power rule vs exponential	x^n differentiates to $n \cdot x^{n-1}$. A constant to the x power is a different rule entirely.
Product rule vs product of derivatives	$d/dx[fg] = f'g + fg'$. Both terms appear. The product of the derivatives alone is always wrong.
Quotient rule signs	Numerator is $f'g - fg'$ with a minus. Denominator is g^2 , squared. Both details matter.
Derivative of $\cos x$	It is $-\sin x$, with the minus sign. The co-functions $\cos$, $\cot$, and $\csc$ all take minus signs in their derivatives.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. If $f(x) = 3x^4 - 2x + 7$, then $f'(x) =$

- A. $12x^3 - 2$
- B. $12x^3 - 2x$
- C. $3x^3 - 2$
- D. $12x^3 - 2 + 7$

2. $d/dx[x^2 \ln x] =$

- A. $2x \ln x + x$
- B. $2x \cdot (1/x)$
- C. $2x \ln x$
- D. $x^2 \cdot (1/x)$

3. $d/dx[\sin x / x] =$

- A. $(x \cos x - \sin x) / x^2$
- B. $(x \cos x + \sin x) / x^2$
- C. $\cos x / 1$
- D. $(\cos x - \sin x) / x$

4. The equation of the tangent line to $f(x) = x^2 + 1$ at $x = 1$ is

- A. $y = 2x$
- B. $y = 2x + 1$
- C. $y - 1 = 2(x - 1)$
- D. $y = x + 1$

Answer Key

- 1. A.** Term by term: $d/dx[3x^4] = 12x^3$ by the constant multiple and power rules, $d/dx[-2x] = -2$, and $d/dx[7] = 0$ by the constant rule. B keeps an x on the -2 term, as if $-2x$ differentiated to $-2x$. C applies the power rule incorrectly to $3x^4$, bringing down nothing. D fails to zero out the constant 7.
- 2. A.** Product rule with $f = x^2$ and $g = \ln x$: $f'g + fg' = 2x \cdot \ln x + x^2 \cdot (1/x) = 2x \ln x + x$. B differentiates both factors and multiplies them, which is the product-of-derivatives mistake. C drops the second term of the product rule. D drops the first term.
- 3. A.** Quotient rule with $f = \sin x$ and $g = x$: $(f'g - fg') / g^2 = (\cos x \cdot x - \sin x \cdot 1) / x^2$. B uses a plus sign in the numerator, the most common quotient rule error. C cancels as if the derivative of a quotient were the quotient of the derivatives. D forgets to square the denominator and mishandles the numerator.
- 4. A.** $f(1) = 1 + 1 = 2$ and $f'(x) = 2x$, so $f'(1) = 2$. Point-slope form gives $y - 2 = 2(x - 1)$, which simplifies to $y = 2x$. B adds an extra 1, confusing the function value with the intercept. C uses the point $(1, 1)$ instead of $(1, 2)$, misreading $f(1)$. D guesses a line through $(0, 1)$ with the wrong slope.

One-Page Recall Check

- State the average rate of change formula and explain what the secant line represents.
- Write the limit definition of the derivative from memory.
- List the four derivative notations and name the Leibniz and Lagrange forms.
- Explain what the difference quotient estimates and what its limit gives.
- State the differentiability-continuity implication and give a counterexample to the converse.
- Name the three ways a continuous function can fail to be differentiable at a point.
- State the power rule and apply it to x^{-3} and $\sqrt{x}$.
- State the constant, sum, difference, and constant multiple rules.
- Give the derivatives of $\sin x$, $\cos x$, e^x , and $\ln x$ from memory.
- State the product rule and use it on $x^3 \cdot \cos x$.
- State the quotient rule and explain why the minus sign and the squared denominator matter.
- Give the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$, and state the sign pattern.
- Find the tangent line to $f(x) = x^3$ at $x = 2$ from scratch.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Differentiation deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Secant line, Tangent line, Derivative of a function, Derivative notation, Difference quotient, Differentiable, Differentiability implies continuity, Power Rule, Constant Rule, Sum Rule, Difference Rule, Constant Multiple Rule, Derivative of $\sin x$, Derivative of $\cos x$, Derivative of e^x , Derivative of $\ln x$, Product Rule, Quotient Rule, Derivative of $\tan x$, Derivative of $\cot x$, Derivative of $\sec x$, Derivative of $\csc x$

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