

Unit 10: Infinite Sequences and Series

Unit 10 asks whether adding infinitely many numbers can produce a finite answer, and how to tell when it does. It covers sequences, the convergence tests for series, the special series every exam expects you to know, and Taylor polynomials and power series for approximating functions.

AP Calculus BC

Infinite Sequences and Series

About 15 minutes to read

How to use this guide

Read it in order the first time because the tests build on each other. Start with what convergence means, learn the geometric and p-series benchmarks, then learn each convergence test with its exact conditions. The exam tests conditions more than computations, so memorize what each test requires before you practice applying it.

After the first read, use the trap boxes and the test-comparison table to review the distinctions the exam tests most. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Unit 10 is about 17 to 18 percent of the AP Calculus BC exam, the largest share of any unit. It is also the unit most students find hardest, partly because each test has conditions that must be stated exactly. A wrong or missing condition on a free-response justification costs the point even when the conclusion is right.

10.1 What Convergence Means

A **sequence** is an ordered list $a_1, a_2, a_3, \dots$ indexed by the natural numbers. A sequence converges when its terms approach a finite value. The sequence $1, 1/2, 1/3, 1/4, \dots$ converges to 0 because the terms get arbitrarily close to 0.

An **infinite series** is a sum $\sum a_n$ over an endless sequence of terms. It may add up to a finite value or fail to settle. The **n th partial sum** S_n is the sum of the series' first n terms, and **convergence of an infinite series** is decided by what happens to these sums. The series $\sum a_n$ has sum S exactly when the limit of its partial sums equals S . Otherwise the series has no sum.

That definition is the foundation everything else rests on. Every test in this unit is a shortcut for answering the same question: do the partial sums settle on a finite value?

Trap. A series and its terms are different objects. The terms of $\sum 1/n$ approach 0, but the series itself diverges. Terms going to zero is necessary for convergence but not enough. Keep the sequence question ("do the terms settle?") separate from the series question ("do the partial sums settle?").

10.2 Geometric Series

A **geometric series** has a constant ratio r between successive terms. It converges exactly when $|r| < 1$, with sum $a/(1 - r)$ for $\sum_{n=0}^{\infty} ar^n$, where a is the first term. It diverges when $|r| \geq 1$.

For example, $\sum_{n=0}^{\infty} (1/3)^n$ has $a = 1$ and $r = 1/3$, so the sum is $1/(1 - 1/3) = 3/2$. When the series starts at $n = 1$ instead, the first term is ar , so $\sum_{n=1}^{\infty} (1/2)^n = (1/2)/(1 - 1/2) = 1$.

Trap. The formula $a/(1 - r)$ assumes the first term of the sum you are evaluating is a . If the series starts at $n = 5$, the first term is ar^5 , not a . Adjust before you plug in.

10.3 The nth Term Test for Divergence

The **nth Term Test for Divergence** says: if $\lim a_n \neq 0$, the series cannot converge. If the terms do not approach zero, the partial sums keep adding a non-shrinking amount and can never settle.

This test only detects divergence. If the limit is zero, the test is inconclusive. The harmonic series $\sum 1/n$ has terms approaching zero yet diverges, which is why this test can never confirm convergence.

Trap. Saying "the terms go to zero, so the series converges by the nth term test" is the most common error in this unit. The test works in one direction only. Terms approaching zero tells you nothing.

10.4 The Integral Test

The **integral test** applies to $\sum a_n$ with positive, decreasing terms coming from a continuous function f where $a_n = f(n)$. The series and the improper integral from 1 to ∞ of f converge or diverge together.

The conditions matter. The terms must be positive, decreasing, and match a continuous function. If the terms are not decreasing, the test does not apply. The conclusion is about convergence only, never about the value of the sum.

10.5 Harmonic Series and p-Series

The **harmonic series** $\sum 1/n$ is the classic example of a series that diverges even though its terms approach zero. It is the reason the nth term test can never prove convergence.

A **p-Series** $\sum 1/n^p$ converges if and only if $p > 1$, and diverges for $p \leq 1$. The harmonic series is the $p = 1$ case. So $\sum 1/n^3$ converges because $3 > 1$, while $\sum 1/\sqrt{n}$ diverges because $p = 1/2 \leq 1$.

Trap. Do not confuse p-series with geometric series. $\sum (1/2)^n$ is geometric with $r = 1/2$ and converges. $\sum 1/n^2$ is a p-series with $p = 2$ and converges. They converge for different reasons, and the exam sometimes presents one disguised as the other.

10.6 The Comparison Tests

The **comparison test** works for series with nonnegative terms. If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. In words, a series smaller than a convergent one converges, and a series larger than a divergent one diverges.

The **limit comparison test** works for positive terms. If $\lim a_n/b_n$ is finite and positive, then $\sum a_n$ and $\sum b_n$ converge or diverge together. The two series behave the same way because their terms are eventually proportional.

For example, $\sum 1/(n^2 - 3n + 1)$ behaves like $\sum 1/n^2$ for large n . The limit of the ratio is 1, which is finite and positive, so the limit comparison test says both converge or both diverge. Since $\sum 1/n^2$ is a convergent p-series, the original series converges.

Trap. Comparison tests need a benchmark series whose behavior you already know, usually geometric or p-series. The comparison itself is the easy part. Choosing the right benchmark is the skill to practice.

10.7 The Alternating Series Test

An **alternating series** is a series whose terms change sign at each step, for example $\sum (-1)^n b_n$ with $b_n > 0$. The sign-flipped harmonic series $\sum (-1)^{n+1}/n$ converges.

The **Alternating Series Test** says a sign-flipping series converges when its positive parts b_n decrease to zero. Both conditions are required: the b_n must be decreasing, and their limit must be zero. The test establishes convergence only, not the sum.

Trap. Forgetting to check that b_n is decreasing is the standard lost point. State both conditions in a free-response justification: the terms decrease, and the limit of the positive parts is zero.

10.8 The Ratio Test

The **ratio test** computes $L = \lim |a_{n+1}/a_n|$. The series converges absolutely if $L < 1$, diverges if $L > 1$, and the test is inconclusive if $L = 1$.

The ratio test is the best choice when the terms involve factorials or exponentials, because the $n+1$ to n ratio simplifies. For $\sum n!/10^n$, the ratio is $(n+1)/10$, which grows without bound, so $L > 1$ and the series diverges. When $L = 1$, try something else. The ratio test says nothing about $\sum 1/n^2$, where the ratio approaches 1 but the series converges.

Trap. The ratio test uses absolute values. That means a conclusion of $L < 1$ gives absolute convergence, which is stronger than plain convergence. Use this when the exam asks about absolute versus conditional convergence.

10.9 Absolute and Conditional Convergence

Absolute convergence means $\sum |a_n|$ has a sum. When a series converges absolutely, the original series $\sum a_n$ has a sum too. **Conditional convergence** means the series settles on a sum while $\sum |a_n|$ does not.

The distinction matters because reordering the terms of a conditionally convergent series can change its sum, while reordering an absolutely convergent series cannot. The alternating harmonic series $\sum (-1)^{n+1}/n$ converges conditionally: it converges by the Alternating Series Test, but $\sum 1/n$ diverges.

10.10 The Alternating Series Error Bound

The **alternating series error bound** says that for a convergent sign-flipping series, stopping at a partial sum leaves an error of at most the first omitted term's magnitude. If you approximate the sum by S_3 , the error is at most a_4 .

For the alternating harmonic series, $S_3 = 1 - 1/2 + 1/3 \approx 0.8333$. The true sum is $\ln 2 \approx 0.6931$, so the actual error is about 0.1402. The bound says the error is at most $1/4 = 0.25$, which holds. The bound is a guarantee, not an estimate of the actual error.

10.11 Taylor Polynomials

A **Taylor polynomial** $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$ is built from a function's derivatives at a . Higher n approximates f better near a . Centered at $a = 0$ it is called a **Maclaurin polynomial**.

For $f(x) = e^x$ centered at 0, the derivatives are all e^x , each equal to 1 at $x = 0$, so $P_3(x) = 1 + x + x^2/2 + x^3/6$.

At $x = 0.2$ this gives 1.221333, while the true value $e^{0.2} \approx 1.221403$. The third-degree polynomial is already accurate to three decimal places.

Trap. A Taylor polynomial approximates the function near the center a . Far from a , the approximation can be terrible. The error bound in the next section quantifies how good the approximation is at a given x .

10.12 The Lagrange Error Bound

The **Lagrange error bound** says $|R_n(x)| \leq M|x - a|^{n+1}/(n+1)!$, where M is a maximum of $|f^{(n+1)}|$ over the interval, and $R_n(x)$ is the remainder after the n th-degree polynomial. For an alternating series, the alternating-series estimate applies instead.

Continuing the e^x example at $x = 0.2$ with $n = 3$: the fourth derivative is e^x , whose maximum on $[0, 0.2]$ is $e^{0.2} \approx 1.2214$, so the error is at most $1.2214 \cdot (0.2)^4/24 \approx 0.0000814$. The actual error was 0.00007, inside the bound.

Trap. M must bound the $(n+1)$ th derivative, not the n th. Using the wrong derivative order is a common error. Also, M has to be a maximum over the whole interval from a to x , not just the value at x .

10.13 Power Series, Radius, and Interval of Convergence

A **power series** $\sum_{n=0}^{\infty} a_n(x - r)^n$ has coefficients a_n and center r . It can represent a function over an interval. The **radius of convergence** R is such that the series converges for $|x - r| < R$ and diverges for $|x - r| > R$. R may be 0 or infinite. It is found with the ratio test.

The **interval of convergence** is the full set of x where the series converges: the open range from the radius, plus any endpoints that pass separate endpoint tests. The ratio test never decides endpoints, so each endpoint gets its own test, often the Alternating Series Test or a p -series comparison.

Term-by-term differentiation and integration means differentiating or integrating a power series one component at a time. The result is a new series with the same radius, representing the derivative or antiderivative of the function. The endpoints may change and need rechecking.

Trap. Radius and interval are different answers. The radius is a number. The interval is a set of x values. A free-response question asking for the interval of convergence expects the endpoints tested and the interval written correctly, not just R .

10.14 and 10.15 Taylor Series, Maclaurin Series, and Building New Series

A **Taylor series** $\sum_{n=0}^{\infty} f^{(n)}(a)/n! \cdot (x - a)^n$ is the infinite-degree extension of the Taylor polynomials. A **Maclaurin series** is the $a = 0$ case.

Four Maclaurin series are the building blocks for constructing others: $1/(1 - x) = \sum x^n$ for $|x| < 1$, $e^x = \sum x^n/n!$, $\sin x = \sum (-1)^n x^{2n+1}/(2n+1)!$, and $\cos x = \sum (-1)^n x^{2n}/(2n)!$.

Representing functions as power series means building a series for a new function from a known one by differentiating, integrating, substituting, or using algebra and geometric-series properties. To get a series for $1/(1 + x^2)$, substitute $-x^2$ for x in the geometric series: $\sum (-x^2)^n = \sum (-1)^n x^{2n}$, valid for $|x| < 1$. Integrating term by term gives a series for $\arctan x$.

Trap. When you substitute into a known series, the interval of convergence transforms with the substitution. Replacing x by $-x^2$ changes $|x| < 1$ into $|x^2| < 1$, which is still $|x| < 1$ here, but check every time.

Which Test to Use

WHAT YOU SEE	TRY FIRST	WHY
Constant ratio between terms	Geometric series	Gives convergence and the actual sum
Terms do not approach zero	nth Term Test	Proves divergence immediately
$1/n^p$ form	p-series	Converges iff $p > 1$
Factorials or exponentials	Ratio test	The $n+1$ to n ratio simplifies
Alternating signs	Alternating Series Test	Check decreasing and limit zero
Looks like a known series	Comparison or limit comparison	Benchmark against geometric or p-series
Positive decreasing terms, integrable form	Integral test	Series and integral agree

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
Sequence vs series	A sequence is a list of terms. A series is a sum of terms. Convergence of one does not imply convergence of the other.
nth term test vs proving convergence	The nth term test only proves divergence. Terms going to zero never proves convergence.
Absolute vs conditional convergence	Absolute means $\sum a_n $ converges, which is stronger. Conditional means the series converges but $\sum a_n $ diverges.
Radius vs interval of convergence	The radius is a number R . The interval is the set of x values, including any endpoints that pass their own tests.
Taylor polynomial vs Taylor series	The polynomial is a finite approximation of degree n . The series is the infinite extension. The Lagrange bound measures the gap between them.
Error bound vs actual error	The bound is a guaranteed maximum. The actual error is usually smaller. Never claim the error equals the bound.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Which of the following series converges?

- A. $\sum_{n=1}^{\infty} 1/n$
- B. $\sum_{n=1}^{\infty} 1/\sqrt{n}$
- C. $\sum_{n=1}^{\infty} 1/n^3$
- D. $\sum_{n=1}^{\infty} n/(n+1)$

2. For the series $\sum_{n=1}^{\infty} (-1)^n/n^2$, which statement is true?

- A. It diverges by the nth term test
- B. It converges conditionally
- C. It converges absolutely
- D. It diverges by the ratio test

3. The third-degree Maclaurin polynomial for f is $P_3(x) = 2 + x - x^2/2 + x^3/3$. What is $f''(0)$?

- A. $-1/2$
- B. -1
- C. 1
- D. 2

4. A power series centered at $x = 2$ converges at $x = 5$ and diverges at $x = -2$. Which of the following must be true?

- A. The radius of convergence is 3
- B. The series converges at $x = -1$
- C. The series diverges at $x = 6$
- D. The interval of convergence is $(-1, 5]$

Answer Key

1. C. This is a p-series with $p = 3 > 1$, so it converges. A is the harmonic series ($p = 1$), which diverges. B is a p-series with $p = 1/2 \leq 1$, which diverges. D has terms $n/(n+1)$ approaching 1, not zero, so it diverges by the nth term test.

2. C. The series of absolute values is $\sum 1/n^2$, a convergent p-series, so the original series converges absolutely. A is wrong because the terms do approach zero. B is wrong because conditional convergence requires $\sum |a_n|$ to diverge, and here it converges. D is wrong because the ratio test gives $L = 1$ here, which is inconclusive, not divergence.

3. B. The Maclaurin polynomial is $P_3(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3!$. Matching coefficients, $f'''(0)/2 = -1/2$, so $f'''(0) = -1$. A forgets to multiply by $2!$. C drops the negative sign. D confuses $f''(0)$ with $f(0) = 2$.

4. C. The center is 2 and the series converges at $x = 5$, which is 3 units away, so $R \geq 3$. It diverges at $x = -2$, which is 4 units away, so $R < 4$. Therefore $3 \leq R < 4$. The point $x = 6$ is 4 units from the center, which is beyond the radius no matter what, so the series must diverge there. A is wrong because R could be larger than 3. B is wrong because if $R = 3$ exactly, $x = -1$ sits on the boundary and needs its own endpoint test. D assumes specific endpoint behavior that is not given.

One-Page Recall Check

- State the definition of series convergence in terms of partial sums.
- Write the geometric series convergence condition and sum formula, and explain the first-term adjustment.
- State the n th Term Test for Divergence and explain why it cannot prove convergence.
- State the three conditions for the integral test.
- State the p -series rule and identify the harmonic series as a special case.
- State both comparison tests and explain when each is easier to use.
- State the two conditions of the Alternating Series Test.
- State the ratio test conclusions for $L < 1$, $L > 1$, and $L = 1$.
- Explain the difference between absolute and conditional convergence.
- State the alternating series error bound and use it on the alternating harmonic series.
- Write the Taylor polynomial formula and the Maclaurin special case.
- State the Lagrange error bound and identify what M must bound.
- Explain the difference between radius and interval of convergence.
- Write the four building-block Maclaurin series from memory.
- Explain how to build a series for $1/(1 + x^2)$ by substitution.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Infinite Sequences and Series deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

[Study this unit in Rycal](#)

Key terms for this unit

Infinite series, Sequence, n th partial sum, Convergence of an infinite series, Geometric series, n th Term Test for Divergence, Integral test, Harmonic series, p -Series, Comparison test, Limit comparison test, Alternating series, Alternating Series Test, Ratio test, Absolute convergence, Conditional convergence, Alternating series error bound, Taylor polynomial, Lagrange error bound, Power series, Radius of convergence, Interval of convergence, Term-by-term differentiation and integration, Taylor series, Maclaurin series, Representing functions as power series.

About this guide. Written for Rycal and aligned to the College Board AP Calculus BC course framework, Unit 10. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.