

Unit 1: Limits and Continuity

Unit 1 is where calculus begins. It introduces rates of change, the limit as the central idea, the tools for evaluating limits, the meaning of continuity, and the Intermediate Value Theorem. Everything in this unit reappears later: the derivative is a limit, improper integrals are limits, and infinite series are limits of partial sums.

AP Calculus BC

Limits and Continuity

About 12 minutes to read

How to use this guide

Work through it in order on the first pass. Rates of change give the limit a reason to exist, the limit gives continuity its definition, and continuity is the hypothesis the Intermediate Value Theorem needs. On the exam, most Unit 1 questions hand you a graph, a table, or an expression and ask what is happening at a point or at infinity.

After reading, use the trap boxes and the comparison tables to drill the distinctions the exam tests most often. Then do the practice questions without looking at the answer key, and finish with the recall check by explaining each item out loud.

What this unit is worth. Limits and Continuity is about 10 to 12 percent of the AP Calculus BC exam.

The unit also matters beyond its weight because every major idea in the course is built on the limit: the derivative in Unit 2, the integral in Unit 6, and the series in Unit 10 all rest on it.

1.1 Introducing Calculus: Can Change Occur at an Instant?

The unit opens with two ways to measure change. The **average rate of change** of a function over an interval is the change in the output divided by the change in the input, written $[f(b) - f(a)] / (b - a)$. On a graph, it is the slope of the **secant line** joining the points $(a, f(a))$ and $(b, f(b))$. For $f(x) = x^3$, the average rate of change over $[0, 2]$ is $(8 - 0) / (2 - 0) = 4$.

The **instantaneous rate of change** is the rate at one point. It is defined as the limit of average rates of change as the interval collapses down to that point. On a graph it is the slope of the **tangent line**, and it is what the derivative computes. For $f(x) = x^3$ at $x = 2$, shrink the interval to $[2, 2 + h]$: the average rate is $[(2 + h)^3 - 8] / h = (12h + 6h^2 + h^3) / h = 12 + 6h + h^2$, which approaches 12 as h approaches 0. So the instantaneous rate at $x = 2$ is 12.

Trap. An average rate over an interval and a rate at a point are different answers. The secant slope 4 above is the average over $[0, 2]$, while the instantaneous rate at $x = 2$ is 12. When a question asks for the rate *at* a point, it wants the limit, not the secant.

1.2 Defining Limits and Using Limit Notation

The **limit of a function**, written $\lim_{x \rightarrow c} f(x) = L$, means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to c , with x not equal to c . Read the notation as "the limit of f of x as x approaches c equals L ." The exam requires correct **limit notation** in justifications, so write the $\lim_{x \rightarrow c}$ part every time you claim something about a limit.

The function need not be defined at c for the limit to exist. The limit says what the function is approaching, not what is happening at the point itself. The limit value and the function value are separate numbers that coincide exactly when the function is continuous there.

Trap. The single most common error in this unit is writing $f(c)$ when the question asked for $\lim_{x \rightarrow c} f(x)$. If the graph has a hole at $x = c$, the limit is still whatever the nearby values approach.

1.3 Estimating Limit Values from Graphs

A **one-sided limit** approaches c from one direction only. From the right it is $\lim_{x \rightarrow c^+} f(x)$; from the left, $\lim_{x \rightarrow c^-} f(x)$. On a graph, follow the curve toward $x = c$ from each side separately and read the y -value each side is heading to. The two-sided limit $\lim_{x \rightarrow c} f(x)$ exists only when both one-sided limits exist and match.

One-sided limits are the natural tool for jump discontinuities. Suppose the left branch of a graph approaches $y = 2$ at $x = 3$ and the right branch approaches $y = 5$. Then $\lim_{x \rightarrow 3^-} f(x) = 2$ and $\lim_{x \rightarrow 3^+} f(x) = 5$, and because the two differ, $\lim_{x \rightarrow 3} f(x)$ does not exist. The plotted value $f(3)$, whatever dot is drawn, changes neither one-sided limit.

Trap. When the two sides disagree, the two-sided limit does not exist, regardless of what $f(c)$ equals. A filled-in dot at $x = c$ sets the function value; it does not create a two-sided limit.

1.5 Determining Limits Using Algebraic Properties of Limits

The **algebraic properties of limits**, or limit laws, let you split a limit apart. The limit of a sum, difference, product, quotient, or power is the matching combination of the individual limits, as long as each limit exists and the denominator's limit is not zero. Constant multiples and powers follow the same pattern.

Direct substitution is almost always the first move, and the limit laws justify it for polynomials and for rational functions wherever the denominator is nonzero. For example, $\lim_{x \rightarrow 1} (x^2 + 2x) / (x + 3) = (1 + 2) / (1 + 3) = 3/4$.

When substitution gives $0/0$, the expression is indeterminate and needs algebra first. Factoring and canceling is the usual fix. For $\lim_{x \rightarrow -2} (x^2 - 4) / (x + 2)$, substitution gives $0/0$, but factoring gives $(x - 2)(x + 2) / (x + 2)$, which equals $x - 2$ for $x \neq -2$, so the limit is -4 .

WHAT SUBSTITUTION GIVES	WHAT TO TRY
A number	Done. That is the limit.
$0/0$	Factor and cancel, or find another algebraic rewrite, then substitute again.
A nonzero number over 0	The function grows without bound. Check each side for an infinite limit and a vertical asymptote.
An oscillating factor like $\cos(1/x)$	Try the Squeeze Theorem from topic 1.8.

Trap. The form $0/0$ is indeterminate, not a final answer. It signals that algebra is hiding the real value, as in the $(x^2 - 4) / (x + 2)$ example where the limit is -4 . Never stop at $0/0$.

1.8 Determining Limits Using the Squeeze Theorem

The **Squeeze Theorem** handles functions that resist direct evaluation. If $g(x) \leq f(x) \leq h(x)$ for x near c , and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$. The two outer functions trap the middle one at the same limit.

The classic application is $\lim_{x \rightarrow 0} x^2 \cdot \cos(1/x)$. The cosine factor oscillates between -1 and 1 forever, so substitution tells you nothing, but $-x^2 \leq x^2 \cdot \cos(1/x) \leq x^2$ for all x , and both bounding functions approach 0 as $x \rightarrow 0$. The squeeze pins the limit at 0 . The same idea proves $\lim_{x \rightarrow 0} \sin x / x = 1$, a result that returns constantly in differentiation and in series.

Trap. The Squeeze Theorem requires both bounding functions to approach the *same* limit. If the bounds head to different values, the theorem says nothing, and the squeezed function could do anything.

1.10 Exploring Types of Discontinuities

A discontinuity is a break in the graph, and there are three kinds. A **removable discontinuity** is a hole: the limit exists at $x = c$, but $f(c)$ is undefined or does not match the limit. Redefining $f(c)$ to equal the limit would repair it. A **jump discontinuity** is where the left- and right-hand limits exist but disagree, so the graph steps from one value to another. A **discontinuity due to a vertical asymptote** is where the function grows without bound as x approaches c , an infinite limit.

TYPE	WHAT YOU SEE	EXAMPLE
Removable	A hole. The limit exists but the point is missing or misplaced.	$f(x) = (x^2 - 1) / (x - 1)$ at $x = 1$. The limit is 2 , but $f(1)$ is undefined.
Jump	A step. The one-sided limits exist and disagree.	A piecewise function with $\lim_{x \rightarrow 1^-} = 2$ and $\lim_{x \rightarrow 1^+} = 5$.
Vertical asymptote	The graph shoots up or down without bound near $x = c$.	$f(x) = 1 / (x - 1)$ at $x = 1$.

Trap. A hole and a vertical asymptote are different breaks. If the limit exists at the point, the discontinuity is removable. If the function grows without bound, it is an infinite discontinuity. Determine the limit first, then classify.

1.11 Defining Continuity at a Point

Continuity at a point rests on three conditions. A function f is continuous at $x = c$ when $f(c)$ is defined, $\lim_{x \rightarrow c} f(x)$ exists, and the two are equal: $\lim_{x \rightarrow c} f(x) = f(c)$. Informally, you can draw through the point without lifting the pencil, and the value you approach matches the value sitting there.

Each discontinuity type breaks the definition in its own way. A removable discontinuity satisfies the limit condition but fails the equality, because $f(c)$ is missing or wrong. A jump discontinuity fails the limit condition outright, since the two sides disagree. An infinite discontinuity fails both, because the function is undefined at c and the limit does not exist as a real number.

Trap. Justification questions expect all three conditions named. Saying "the function is defined at $x = 1$ " is not enough. State that $f(1)$ is defined, the limit exists, and the limit equals $f(1)$, with the actual values shown.

1.12 Confirming Continuity over an Interval

Continuity over an interval means the function is continuous at every point of the interval. For interior points, that is the three-condition test from 1.11. At an endpoint of a closed interval, only the relevant one-sided limit must equal the function value, since there is no second side to check.

This is why the familiar function families are straightforward. Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined, which is everywhere the denominator is nonzero. Sums, differences, products, and quotients of continuous functions remain continuous wherever the result is defined, so most expressions on the exam are continuous on their domains by construction.

Trap. "Continuous on its domain" does not mean "continuous everywhere." The function $f(x) = 1/x$ is continuous on its domain, but it is not continuous at $x = 0$, because 0 is not in its domain. Read carefully which claim the question is making.

1.14 Connecting Infinite Limits and Vertical Asymptotes

An **infinite limit** describes a function growing without bound near a point, written $\lim_{x \rightarrow c} f(x) = \infty$ or $-\infty$. Strictly speaking the limit does not exist as a real number, but the infinity notation records the behavior, and the exam expects you to use it. For $f(x) = 1 / (x - 1)^2$, the values blow up to positive infinity from both sides, so $\lim_{x \rightarrow 1} f(x) = \infty$.

A **vertical asymptote** is the line $x = c$, and it appears exactly when f has an infinite limit as x approaches c from at least one side. One side suffices. The sign can differ by side, so check each one. For $f(x) = 1/(x - 1)$, $\lim_{x \rightarrow 1^+} 1/(x - 1) = \infty$ while $\lim_{x \rightarrow 1^-} 1/(x - 1) = -\infty$, and $x = 1$ is a vertical asymptote either way.

QUESTION BEING ASKED	WHAT YOU ARE FINDING
$\lim_{x \rightarrow c} f(x)$ with x finite	A value, a one-sided split, or an infinite limit at a point. May give a vertical asymptote.
$\lim_{x \rightarrow \infty} f(x)$ with x unbounded	End behavior. May give a horizontal asymptote. See topic 1.15.

Trap. Infinite limits and limits at infinity are different ideas wearing similar notation. The first is about x approaching a finite number while the function blows up. The second is about x itself growing without bound. Match the question to the right row before you start.

1.15 Connecting Limits at Infinity and Horizontal Asymptotes

A **limit at infinity**, $\lim_{x \rightarrow \infty} f(x) = L$, means $f(x)$ approaches L as x grows without bound. A **horizontal asymptote** is the line $y = L$, and it appears when $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$. **End behavior** describes what happens in both directions, $x \rightarrow \infty$ and $x \rightarrow -\infty$, often summarized by the horizontal asymptotes.

For rational functions, compare degrees. If the top degree is smaller, the limit is 0. If the degrees are equal, divide by the highest power and read the ratio of leading coefficients: $\lim_{x \rightarrow \infty} (5x^2 + 2) / (x^2 - 3x) = \lim_{x \rightarrow \infty} (5 + 2/x^2) / (1 - 3/x) = 5/1 = 5$, so $y = 5$ is a horizontal asymptote. If the top degree is larger, the function grows without bound and there is no horizontal asymptote.

Trap. A graph may cross its horizontal asymptote. The asymptote describes only where the function settles as x gets very large. A question asking whether the graph crosses $y = 5$ at some finite x is testing something the asymptote cannot answer.

1.16 Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** says a continuous function cannot skip values. If f is continuous on the closed interval $[a, b]$, and N is any value between $f(a)$ and $f(b)$, then there is at least one c in the open interval (a, b) with $f(c) = N$. The hypotheses matter: the interval must be closed, the function must be continuous on it, and N must lie between the endpoint values.

A standard justification: $k(x) = x^3 - 3x + 1$ is a polynomial, hence continuous everywhere, in particular on $[0, 1]$. Since $k(0) = 0 - 0 + 1 = 1$ and $k(1) = 1 - 3 + 1 = -1$, and 0 lies between 1 and -1 , the IVT guarantees some c in $(0, 1)$ with $k(c) = 0$. The theorem promises that such a c exists, not where it is.

Trap. The IVT needs every one of its hypotheses. If the function is not continuous on $[a, b]$, or if N is not between $f(a)$ and $f(b)$, the conclusion does not follow. The converse is also false: a discontinuous function can still hit every intermediate value, but the IVT is not what guarantees it.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
$\lim_{x \rightarrow c} f(x)$ vs $f(c)$	The limit is what the function approaches. $f(c)$ is what is actually there. They agree only when f is continuous at c .
One-sided vs two-sided limit	The two-sided limit exists only when both one-sided limits exist and agree. A jump kills the two-sided limit.
Removable vs jump discontinuity	Removable means the limit exists and the point is the problem. Jump means the limit itself does not exist.
Infinite limit vs limit at infinity	Infinite limit: x approaches a finite c and f blows up, giving a vertical asymptote. Limit at infinity: x grows without bound, giving end behavior and possibly a horizontal asymptote.
IVT hypotheses vs conclusion	Continuity on $[a, b]$ plus N between $f(a)$ and $f(b)$ gives existence of c . Drop a hypothesis and the guarantee is gone.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. $\lim_{x \rightarrow -2} (x^2 - 4) / (x + 2) =$

- A. 0
- B. -4
- C. Does not exist
- D. 4

2. The graph of f has a jump at $x = 3$. The left branch approaches $y = 2$, the right branch approaches $y = 5$, and $f(3) = 5$. What is $\lim_{x \rightarrow 3} f(x)$?

- A. 2
- B. 5
- C. Does not exist
- D. 3

3. Let $f(x) = (x^2 - 1) / (x - 1)$ for $x \neq 1$, and $f(1) = 3$. Which statement is true?

- A. f is continuous at $x = 1$
- B. f has a removable discontinuity at $x = 1$
- C. f has a jump discontinuity at $x = 1$
- D. f has a vertical asymptote at $x = 1$

4. If $-2x^2 \leq h(x) \leq 2x^2$ for all x near 0, then $\lim_{x \rightarrow 0} h(x) =$

- A. 0
- B. Does not exist
- C. 2
- D. Cannot be determined from this information

5. $\lim_{x \rightarrow 1^+} 1 / (x - 1) =$

- A. ∞
- B. $-\infty$
- C. 0
- D. Does not exist

6. $\lim_{x \rightarrow \infty} (5x^2 + 2) / (x^2 - 3x) =$

- A. 5
- B. 0
- C. ∞
- D. $5/3$

7. The function f is continuous on $[1, 4]$, with $f(1) = -2$ and $f(4) = 7$. Which of the following must be true?

- A. $f(2.5) = 2.5$
- B. There is a value c in $(1, 4)$ such that $f(c) = 3$
- C. f has a zero at $x = 2.5$
- D. f is differentiable on $(1, 4)$

8. The graph of f has a horizontal asymptote $y = -2$ as $x \rightarrow -\infty$. Which of the following must be true?

- A. $\lim_{x \rightarrow -\infty} f(x) = -2$
- B. $f(-2) = -2$
- C. The graph of f never crosses the line $y = -2$
- D. $\lim_{x \rightarrow -2} f(x) = \infty$

Answer Key

- 1. B.** Direct substitution gives $0/0$, so factor: $(x - 2)(x + 2) / (x + 2) = x - 2$ for $x \neq -2$, and the limit is -4 . A stops at the indeterminate form $0/0$ and treats it as an answer. C confuses "indeterminate" with "does not exist"; the algebra reveals a finite limit. D changes the sign of the correct value.
- 2. C.** The one-sided limits are 2 from the left and 5 from the right. They disagree, so the two-sided limit does not exist. A reports only the left-hand limit. B reports the function value $f(3)$, which matches the right-hand limit here but is not the two-sided limit. D confuses the x -value where the jump happens with a y -value.
- 3. B.** Factoring gives $(x - 1)(x + 1) / (x - 1) = x + 1$ for $x \neq 1$, so the limit is 2, but $f(1) = 3$. The limit exists while the function value disagrees, which is the definition of a removable discontinuity. A ignores the mismatch between the limit and $f(1)$. C is wrong because a jump requires disagreeing one-sided limits, and here both sides approach 2. D is wrong because the limit is finite; a vertical asymptote needs an infinite limit.
- 4. A.** Both bounds approach 0 as $x \rightarrow 0$: $\lim_{x \rightarrow 0} 2x^2 = 0$ and $\lim_{x \rightarrow 0} (-2x^2) = 0$. The Squeeze Theorem forces $h(x)$ to 0 as well. B ignores the theorem entirely. C may come from misreading $2x^2$ near 0 as approaching 2. D is the trap of thinking the bounds are not enough; matching bounds are exactly what the theorem needs.
- 5. A.** From the right, $x - 1$ is a small positive number, so $1 / (x - 1)$ is a large positive number: the limit is ∞ . B ignores the side and reports the left-hand behavior instead. C confuses an infinite limit with a limit at infinity of $1/x$. D is tempting because infinite limits do not exist as real numbers, but the question asks for the one-sided behavior, which is precisely ∞ .
- 6. A.** The degrees are equal, so divide by x^2 : $(5 + 2/x^2) / (1 - 3/x)$, which approaches $5/1 = 5$. B would be correct only if the denominator degree were larger. C treats the leading term as dominating without checking that the degrees match. D takes the ratio of the constant terms, which is meaningless here since the x^2 terms control the behavior.
- 7. B.** The IVT applies: f is continuous on $[1, 4]$ and 3 lies between $f(1) = -2$ and $f(4) = 7$, so some c in $(1, 4)$ satisfies $f(c) = 3$. A and C both claim the theorem locates the value; it only guarantees existence, never the position. D confuses continuity with differentiability, which the IVT never requires.
- 8. A.** A horizontal asymptote $y = -2$ as $x \rightarrow -\infty$ is exactly the statement that $\lim_{x \rightarrow -\infty} f(x) = -2$. B confuses the asymptote's y -value with a function value at $x = -2$. C is the classic trap: a graph may cross its horizontal asymptote at finite x values; the asymptote only governs far-out behavior. D confuses a horizontal asymptote with a vertical one.

One-Page Recall Check

- State the average rate of change formula and explain what the secant line represents.
- Explain how the instantaneous rate of change is defined as a limit of average rates.
- Write $\lim_{x \rightarrow c} f(x) = L$ in words, and explain why $f(c)$ need not exist.
- State the condition for a two-sided limit to exist in terms of one-sided limits.
- List the limit laws and the two conditions needed to use the quotient law.
- Evaluate $\lim_{x \rightarrow -2} (x^2 - 4) / (x + 2)$ from scratch, showing the algebra.
- State the Squeeze Theorem and use it to find $\lim_{x \rightarrow 0} x^2 \cdot \cos(1/x)$.
- Describe each discontinuity type and name which continuity condition it violates.
- State the three conditions for continuity at a point.
- Explain what changes at an endpoint when checking continuity over a closed interval.
- Explain how a vertical asymptote follows from an infinite limit on at least one side.
- Give the degree rules for limits at infinity of rational functions.
- State the Intermediate Value Theorem with all of its hypotheses.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Limits and Continuity deck under AP Calculus BC. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Average rate of change, Instantaneous rate of change, Limit of a function, Limit notation, One-sided limit, Algebraic properties of limits (limit laws), Squeeze Theorem, Removable discontinuity, Jump discontinuity, Discontinuity due to a vertical asymptote, Continuity at a point, Continuity over an interval, Infinite limit, Vertical asymptote, Limit at infinity, Horizontal asymptote, End behavior, Intermediate Value Theorem (IVT).

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