

Unit 8: Applications of Integration

Unit 8 puts the definite integral to work: average values, motion problems, net change, areas between curves, and volumes by cross sections, discs, and washers. CED topics 8.1, 8.2, 8.3, 8.4, 8.5, 8.7, 8.9, and 8.11.

AP Calculus AB

Applications of Integration

About 15 minutes to read

How to use this guide

Read it in order the first time because the setups build on each other. Average value and net change come first, then area between curves, then the three volume methods. Every section follows the same exam skill: translating a geometric or physical description into the correct integral. The trap boxes name the wrong setups students reach for most often.

After the first read, use the tables to review which method fits which situation. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is really about. Every question in this unit is a setup question. The exam rarely asks you to evaluate a hard integral here. It asks whether you can look at a region, a velocity graph, or a rate of change and write the right integral for the quantity being asked. If your setup is right, the algebra that follows is usually short.

8.1 Finding the Average Value of a Function on an Interval

The **average value of a function** f over $[a, b]$ is $(1/(b-a)) \cdot \int_a^b f(x) dx$. Think of it as the height of a rectangle over $[a, b]$ that has the same area as the area under the curve. The $(1/(b-a))$ factor is what turns a total area into an average.

Example. The average value of $f(x) = x^2$ on $[0, 3]$ is $(1/3) \int_0^3 x^2 dx = (1/3)[x^3/3]_0^3 = (1/3)(9) = 3$.

Trap. The most common miss is dropping the $1/(b-a)$ factor and reporting the value of the integral as the answer. The integral alone gives total area. The average always divides by the width of the interval.

8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals

Velocity is the rate of change of position, so integrating velocity gives position back. **Displacement** is the net change in position over $[a, b]$: $\int_a^b v(t) dt$. It can be negative, because moving backward cancels moving forward.

Total distance traveled is the full path length: $\int_a^b |v(t)| dt$. Absolute value forces every part of the trip to count positively. Total distance is never negative.

Example. For $v(t) = t - 2$ on $[0, 4]$, velocity is negative on $[0, 2)$ and positive on $(2, 4]$. Displacement is $\int_0^4 (t - 2) dt = [t^2/2 - 2t]_0^4 = 0$, so the particle ends where it started. Total distance is $\int_0^2 (2 - t) dt + \int_2^4 (t - 2) dt = 2 + 2 = 4$.

Trap. When velocity changes sign, displacement and distance disagree. Read the question. If it asks for displacement or net change, integrate $v(t)$ directly. If it asks for total distance, split the integral at every zero of $v(t)$ and add the absolute values of the pieces.

QUANTITY	INTEGRAL	SIGN
Displacement	$\int_a^b v(t) dt$	Can be negative
Total distance traveled	$\int_a^b v(t) dt$	Never negative
Change in velocity	$\int_a^b a(t) dt$	Can be negative

8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts

A rate of change accumulates into a total amount. **Net change** is the integral of the rate: $\int_a^b f'(x) dx = f(b) - f(a)$. The integral of a rate tells you how much the quantity changed, not how much is there at the end.

Example. Water enters a tank at $r(t) = 4t$ gallons per minute for $0 \leq t \leq 5$, and the tank starts with 10 gallons. The net change is $\int_0^5 4t dt = [2t^2]_0^5 = 50$ gallons, so the final amount is $10 + 50 = 60$ gallons.

Trap. Net change is not the final amount. If the problem gives an initial value, add the integral to it. The integral answers "how much did it change by." The final amount is initial value plus net change.

8.4 Finding the Area Between Curves Expressed as Functions of x

For $f \geq g$ on $[a, b]$, the **area between the curves** is $\int_a^b [f(x) - g(x)] dx$. Top curve minus bottom curve, integrated across the interval. Find where the curves cross first, because those intersection points are your limits of integration.

Example. The region bounded by $y = x^2$ and $y = x$ lies between $x = 0$ and $x = 1$, where $x \geq x^2$. Its area is $\int_0^1 (x - x^2) dx = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6$.

Trap. Two misses show up here. First, subtracting in the wrong order gives a negative area. Second, when the curves cross more than once, one integral across the whole span is wrong. Split at every intersection point and set up top minus bottom on each piece.

8.5 Finding the Area Between Curves Expressed as Functions of y

Some regions are easier to slice horizontally. If $x = f(y)$ is to the right of $x = g(y)$ on $[c, d]$, the **area between the curves** is $\int_c^d [f(y) - g(y)] dy$. Right curve minus left curve, integrated with respect to y .

Example. The region bounded by $x = y$ and $x = y^2$ lies between $y = 0$ and $y = 1$, where $y \geq y^2$. Its area is $\int_0^1 (y - y^2) dy = 1/6$. This is the same region as the 8.4 example, computed with horizontal slices instead of vertical ones.

Trap. Choose the variable that keeps the top and bottom (or right and left) roles consistent. If the curves trade places partway through in x , try y . The limits must match the variable you integrate with respect to. Bounds in x with a dy integral is a setup error.

8.7 Volumes with Cross Sections: Squares and Rectangles

Some solids are built by stacking flat shapes. If cross sections perpendicular to the x -axis have area $A(x)$, the **volume with cross sections** is $\int_a^b A(x) \, dx$. The work is writing $A(x)$ from the shape: squares give $A = s^2$, rectangles give $A = l \cdot w$, where the side lengths come from the given functions.

Example. Take the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$. Cross sections perpendicular to the x -axis are squares with side $s = \sqrt{x}$. Then $A(x) = (\sqrt{x})^2 = x$, and the volume is $\int_0^4 x \, dx = [x^2/2]_0^4 = 8$.

Trap. The integral needs the area of the cross section, not its side length. Squaring happens when you write $A(x)$, before you integrate. Integrating the side length s instead of s^2 is the standard miss on this topic.

8.9 Volume with Disc Method: Revolving Around the x - or y -Axis

Revolving a region around an axis stacks circular discs. The **disc method** volume is $V = \pi \int_a^b [R(x)]^2 \, dx$, where $R(x)$ is the radius from the curve to the axis of revolution. Each disc has area πR^2 , and the integral adds them up.

Example. Revolve the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ around the x -axis. The radius is $R(x) = \sqrt{x}$, so $V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = 8\pi$. Compare with the 8.7 example: same region, but revolving it into discs gives 8π while stacking squares gave 8.

Trap. The radius must be squared before integrating. $\pi \int R \, dx$ is wrong. The disc area is πR^2 , so the integrand is always the square of the radius.

8.11 Volume with Washer Method: Revolving Around the x- or y-Axis

When the region does not touch the axis of revolution, the solid has a hole, and each slice is a washer: a disc with a disc removed. The **washer method** volume is $V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$, where R is the outer radius and r the inner radius.

Example. Revolve the region between $y = x + 2$ and $y = x$ on $[0, 3]$ around the x-axis. The outer radius is $R = x + 2$ and the inner radius is $r = x$. Then $V = \pi \int_0^3 [(x + 2)^2 - x^2] dx = \pi \int_0^3 (4x + 4) dx = \pi [2x^2 + 4x]_0^3 = 30\pi$.

Trap. Square each radius first, then subtract: $\pi \int (R^2 - r^2)$. The common error is subtracting first and squaring second, $\pi \int (R - r)^2$, which computes the volume of a disc with radius equal to the gap. That is a different and smaller solid.

Choosing the volume method

SITUATION	METHOD	INTEGRAND
Region touches the axis of revolution	Disc (8.9)	πR^2
Region leaves a gap from the axis	Washer (8.11)	$\pi(R^2 - r^2)$
Slices are squares or rectangles	Cross sections (8.7)	$A(x)$, e.g. s^2

Confusions That Cost Points

PAIR	WHAT SEPARATES THEM
Average value vs. integral value	The integral gives total area. Average value divides by $(b - a)$.
Displacement vs. total distance	Displacement is $\int v \, dt$ and can be negative. Distance is $\int v \, dt$ and never is.
Net change vs. final amount	The integral of a rate is the change. Add the initial value for the final amount.
Area with dx vs. area with dy	Vertical slices use top minus bottom with x bounds. Horizontal slices use right minus left with y bounds.
Disc vs. washer	No gap to the axis means disc. A gap means washer with an inner radius.
$R^2 - r^2$ vs. $(R - r)^2$	Washers square each radius first. Squaring the difference gives the wrong solid.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. What is the average value of $f(x) = 4x^3$ on the interval $[0, 2]$?

- A. 8
- B. 16
- C. 4
- D. 32

2. A particle moves along a line with velocity $v(t) = t^2 - 4t + 3$ for $0 \leq t \leq 4$. What is the total distance traveled by the particle?

- A. $4/3$
- B. $8/3$
- C. 4
- D. 0

3. Water enters a tank at a rate of $r(t) = 20 - 2t$ gallons per minute for $0 \leq t \leq 10$. The tank initially holds 15 gallons. How many gallons does it hold after 10 minutes?

- A. 100
- B. 115
- C. 200
- D. 15

4. What is the area of the region enclosed by the graphs of $y = 4 - x^2$ and $y = x + 2$?

- A. $-9/2$
- B. $1/2$
- C. $27/2$
- D. $9/2$

5. What is the area of the region enclosed by the graphs of $x = y^2$ and $x = y + 2$?
- A. $-9/2$
 - B. $9/2$
 - C. $10/3$
 - D. $1/2$
6. The base of a solid is the region under $y = 2x$ from $x = 0$ to $x = 3$. Cross sections of the solid perpendicular to the x -axis are squares whose sides lie in the base region. What is the volume of the solid?
- A. 9
 - B. 18
 - C. 36
 - D. 108
7. The region bounded by $y = 2x$, the x -axis, and $x = 1$ is revolved about the x -axis. What is the volume of the resulting solid?
- A. π
 - B. $2\pi/3$
 - C. $8\pi/3$
 - D. $4\pi/3$
8. The region between $y = x + 1$ and $y = x$ on $[0, 2]$ is revolved about the x -axis. What is the volume of the resulting solid?
- A. 2π
 - B. 6π
 - C. $26\pi/3$
 - D. $8\pi/3$

Answer Key

- 1. A.** The average value is $(1/2)\int_0^2 4x^3 dx = (1/2)[x^4]_0^2 = 16/2 = 8$. B gives the integral 16 without dividing by the interval width, the classic average-value trap. C divides by 4 instead of by $(b - a) = 2$. D doubles the integral instead of dividing.
- 2. C.** Factor: $v(t) = (t - 1)(t - 3)$, so velocity changes sign at $t = 1$ and $t = 3$. Total distance is $\int_0^1 v dt + |\int_1^3 v dt| + \int_3^4 v dt = 4/3 + 4/3 + 4/3 = 4$. A is the displacement $\int_0^4 v dt = 4/3$, the trap for readers who skip the word total. B adds only two of the three pieces. D confuses the particle returning near its start with traveling no distance.
- 3. B.** Net change is $\int_0^{10} (20 - 2t) dt = [20t - t^2]_0^{10} = 100$ gallons. Final amount is initial plus net change: $15 + 100 = 115$. A reports the net change and forgets the initial 15 gallons. C integrates only the constant term 20. D repeats the initial amount without any accumulation.
- 4. D.** The curves intersect where $4 - x^2 = x + 2$, giving $x = -2$ and $x = 1$, with $4 - x^2$ on top. Area is $\int_{-2}^1 [(4 - x^2) - (x + 2)] dx = [2x - x^2/2 - x^3/3]_{-2}^1 = 9/2$. A subtracts bottom minus top, producing a negative area. B integrates over the wrong interval. C comes from an arithmetic slip that triples the correct value.
- 5. B.** The curves intersect where $y^2 = y + 2$, giving $y = -1$ and $y = 2$, with $x = y + 2$ on the right. Area is $\int_{-1}^2 [(y + 2) - y^2] dy = [y^2/2 + 2y - y^3/3]_{-1}^2 = 9/2$. A subtracts left minus right. C evaluates only the upper-limit portion of the antiderivative. D uses the wrong bounds.
- 6. C.** Each cross section is a square with side $s = 2x$, so its area is $A(x) = (2x)^2 = 4x^2$. Volume is $\int_0^3 4x^2 dx = 4[x^3/3]_0^3 = 36$. A integrates the side length $2x$ instead of the area, skipping the squaring step. B squares only the coefficient, using $A = 2x^2$. D cubes the side, treating the slice as a cube.
- 7. D.** The radius is $R = 2x$, so $V = \pi \int_0^1 (2x)^2 dx = \pi \int_0^1 4x^2 dx = 4\pi/3$. A integrates the unsquared radius, $\pi \int_0^1 2x dx$. B uses $R^2 = 2x^2$, squaring only the variable. C doubles the correct integrand.
- 8. B.** The outer radius is $R = x + 1$ and the inner radius is $r = x$, so $V = \pi \int_0^2 [(x + 1)^2 - x^2] dx = \pi \int_0^2 (2x + 1) dx = \pi [x^2 + x]_0^2 = 6\pi$. A uses the 1-unit gap as the radius, computing a disc of thickness 1 instead of a washer. C computes only the outer disc, forgetting to subtract the hole. D computes only the inner disc, the hole itself.

When you check your answers, note which setup each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Applications of Integration deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Applications of Integration deck and let spaced review bring them back over the next few days.

- Write the average value formula and explain what the $1/(b - a)$ factor does.
- State the integrals for displacement and total distance, and say when they disagree.
- Explain how to handle a velocity function that changes sign on the interval.
- Define net change and relate it to $f(b) - f(a)$.
- Solve a tank problem: given a rate and an initial amount, find the final amount.
- Set up the area between two curves as functions of x , including how to find the limits.
- Set up the area between two curves as functions of y , including how to find the limits.
- Explain when to use dx and when to use dy for an area problem.
- Write the cross-section volume formula and explain how $A(x)$ is built from the shape.
- Write the disc method formula and identify the radius in a worked example.
- Write the washer method formula and explain why each radius is squared before subtracting.
- Choose disc, washer, or cross sections for a region that touches the axis, one with a gap, and one with square slices.
- Explain the difference between $\pi \int (R^2 - r^2)$ and $\pi \int (R - r)^2$.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Applications of Integration deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Average value of a function, displacement, total distance traveled, net change, area between curves (functions of x), area between curves (functions of y), volume with cross sections, disc method, washer method.

About this guide. Written for Rycal and aligned to the College Board AP Calculus AB course framework, Unit 8. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.