

Unit 7: Differential Equations

Unit 7 covers differential equations: what they model, how to verify solutions, slope fields, separation of variables, initial conditions, and exponential models. CED topics 7.1, 7.2, 7.3, and 7.6 through 7.8.

AP Calculus AB

Differential Equations

About 10 minutes to read

How to use this guide

Read it in order the first time because the topics form a sequence. A differential equation describes how a quantity changes, verifying checks a proposed solution, a slope field pictures all the possibilities at once, separation of variables finds the whole family of solutions, an initial condition picks out the one you need, and the exponential model is the most common case. Exam questions usually test exactly one of these steps.

After the first read, use the trap boxes and the table to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

Why this unit feels different. Most of calculus starts with a function and asks for its rate of change. This unit reverses the direction. You start with a description of how a quantity changes and rebuild the function from that description. Every question type here is one step of that rebuild: check a candidate, picture the possibilities, solve in general, then pin down the single solution you need.

7.1 Modeling Situations with Differential Equations

A **differential equation** is an equation that relates a function to its derivatives. The example in the course is $dy/dx = ky$. It models how a quantity changes: the left side is the rate of change, and the right side says what that rate depends on.

The habit to build is reading the equation as a sentence. $dy/dx = 2x$ says the slope at any point depends only on x . $dy/dx = ky$ says the rate of change is proportional to the current value of y itself. $dy/dx = xy$ says the rate depends on both. Modeling a situation means translating a verbal description into one of these sentences. "A population grows at a rate proportional to its size" becomes $dy/dt = ky$, with t for time. "Water drains from a tank at a rate proportional to the square root of the depth" becomes $dh/dt = -k\sqrt{h}$.

Trap. A differential equation must contain a derivative. An equation like $y = 3x^2 - 1$ is a function, not a differential equation, and a statement like $y(2) = 11$ is a single value. Questions that ask you to identify the differential equation are testing whether you look for dy/dx or y' in the expression.

7.2 Verifying Solutions for Differential Equations

A **solution of a differential equation** is a function that satisfies the equation when substituted in, along with its derivatives. Verifying is a check, not a new computation. Take the proposed function, differentiate it, plug both the function and its derivative into the differential equation, and see whether both sides match.

Check whether $y = 5e^{-2x}$ is a solution of $dy/dx = -2y$. Differentiate: $dy/dx = -10e^{-2x}$. Substitute: the left side is $-10e^{-2x}$ and the right side is $-2(5e^{-2x}) = -10e^{-2x}$. Both sides match, so it is a solution. Now check $y = x^2 + 1$ against $dy/dx = 2x$. Differentiating gives $dy/dx = 2x$, which matches the right side exactly, so this is also a solution. Notice that $y = x^2$ would have worked too. A differential equation usually has many solutions, which is the point of the general solution in 7.6.

Trap. A function that satisfies the equation at one point is not enough. The two sides must match as expressions, for the variable values in the domain, not just at a single x . Also, verifying requires substituting both y and dy/dx . Plugging in y alone and declaring victory skips the derivative check the definition requires.

7.3 Sketching Slope Fields

A **slope field** is a grid of short line segments showing the slope dy/dx at many points. Each segment is a tiny piece of a possible solution curve. A solution curve drawn through the field follows the segments, staying tangent to them as it goes.

To draw one segment, pick a point (a, b) , evaluate the differential equation there, and draw a short segment through (a, b) with that slope. For $dy/dx = x/2$, the point $(2, 1)$ gets a segment of slope 1, and the point $(-2, 5)$ gets a segment of slope -1. Along the line $x = 0$ the slope is 0 everywhere, so every segment there is horizontal. The field then shows the whole family of solutions at a glance, before any solving happens.

Reading a slope field is the reverse skill. Find where segments are horizontal ($dy/dx = 0$ there), where they tilt upward to the right (positive slope), and where they tilt downward (negative slope). A proposed solution curve must stay tangent to the segments it passes through. If a curve crosses segments at an angle, it is not a solution.

Trap. The slope of a segment comes from plugging the point into the differential equation, and students mix up which variable matters. For $dy/dx = y$, the slope depends on the y -coordinate, so segments are horizontal along $y = 0$, not along $x = 0$. Read the right side of the equation to see which coordinate controls the slope.

Trap. A horizontal segment means the slope is zero at that point, not that the function value is zero. For $dy/dx = y - 2$, segments are horizontal where $y = 2$, and the solution curves there are flat, but y itself equals 2, not 0.

7.6 Finding General Solutions Using Separation of Variables

Separation of variables solves differential equations of the form $dy/dx = g(x) \cdot h(y)$. The move is algebraic: rewrite as $(1/h(y)) dy = g(x) dx$, so each side involves only one variable, then integrate both sides.

Solve $dy/dx = 2xy$. Divide both sides by y to separate: $(1/y) dy = 2x dx$. Integrate: $\int (1/y) dy = \int 2x dx$, which gives $\ln|y| = x^2 + C$. Exponentiate both sides: $|y| = e^{x^2 + C} = e^C \cdot e^{x^2}$. Writing A for the constant e^C gives $y = Ae^{x^2}$. Check it: $dy/dx = 2x \cdot Ae^{x^2} = 2x \cdot y$, which matches. This is a **general solution**, the family of all solutions of the differential equation, containing an arbitrary constant.

One constant is enough. When you integrate both sides you get a constant on each side, but two arbitrary constants merge into one, so write a single C . Also, the step that divides by $h(y)$ assumes $h(y)$ is not zero. For $dy/dx = 2xy$, dividing by y assumes $y \neq 0$, and indeed $y = 0$ is a solution the separated form cannot produce. The exam rarely asks about it, but knowing why the division needs the assumption keeps the algebra honest.

Trap. The most common algebra error is mishandling the constant. Writing C on both sides and solving as if they were different numbers, or dropping C entirely, both break the solution. One C , kept through the whole computation, gives the full family. A second common error is integrating $(1/y) dy$ to $y^2/2$ or forgetting the absolute value. The antiderivative of $1/y$ is $\ln|y|$.

7.7 Finding Particular Solutions Using Initial Conditions and Separation of Variables

A **particular solution** is the single solution of a differential equation that also satisfies a given **initial condition**, a specified value of the solution such as $y(0) = 3$. The initial condition determines the constant C in the general solution, so no arbitrary constant remains.

The order of work matters. First find the general solution by separation of variables, then use the initial condition to solve for C . For $dy/dx = 2xy$ with $y(0) = 4$: the general solution from 7.6 is $y = Ae^{x^2}$. Substitute $x = 0$ and $y = 4$: $4 = Ae^0 = A$, so $A = 4$. The particular solution is $y = 4e^{x^2}$. As a second example, $dy/dx = 2x + 1$ with $y(0) = 5$ gives the general solution $y = x^2 + x + C$, and $y(0) = C = 5$, so the particular solution is $y = x^2 + x + 5$.

Notice what the initial condition does not do: it does not change the shape of the solution family. It selects one curve from the family, the one passing through the given point. On a slope field, the initial condition picks the starting point, and the particular solution is the curve through the field that passes through it.

Trap. Plugging the initial condition into the differential equation itself, instead of into the general solution, is the standard wrong move. The initial condition is a point on the solution curve, so it belongs in the solved form $y = \dots$, where it pins down C . Also, substitute the full ordered pair. For $y(1) = 3$, $x = 1$ goes into the exponent too, not just C .

7.8 Exponential Models with Differential Equations

An **exponential model** describes a quantity whose rate of change is proportional to the quantity itself: $dy/dt = ky$, with solutions $y = Ce^{kt}$. It grows when $k > 0$ and decays when $k < 0$. The constant C is the starting value, since $y(0) = Ce^0 = C$.

A bacteria culture satisfies $dy/dt = 0.3y$ with $y(0) = 100$. The model gives $y = 100e^{0.3t}$. At $t = 10$ the population is $100e^3$. Here $k = 0.3$ is the relative growth rate, 30 percent per unit time, not an amount added each step. For decay, a sample with $dy/dt = -0.05y$ and $y(0) = 60$ gives $y = 60e^{-0.05t}$, shrinking toward zero as t grows.

To find when a decaying quantity halves, set the model equal to half the start and solve. For $dy/dt = -0.04y$ with $y(0) = 120$: $60 = 120e^{-0.04t}$, so $1/2 = e^{-0.04t}$. Take logs: $\ln(1/2) = -0.04t$, and since $\ln(1/2) = -\ln 2$, $t = (\ln 2)/0.04$. The same pattern works for any fraction. Doubling time for growth uses $2 = e^{kt}$ the same way.

Trap. The sign of k is the whole story, and questions flip it. In $dy/dt = ky$, a positive k means growth and a negative k means decay, regardless of the words around the problem. Also, k is a rate, not a multiplier on t alone. In $y = 50e^{0.2t}$, the value at $t = 10$ is $50e^2$, because the exponent is $kt = 0.2 \cdot 10$, not k by itself.

Trap. Exponential change is not linear change. A quantity with $dy/dt = 0.2y$ does not gain the same amount each year. It gains 20 percent of its current value, so the yearly gain itself grows. Any option that adds a fixed amount per time step is describing a different model.

Confusions That Cost Points

QUESTION TYPE	WHAT TO DO
Identify or write the differential equation	Look for a derivative in the equation. Translate "rate of change" language into dy/dx or dy/dt equals something.
Verify a proposed solution	Differentiate the candidate, substitute y and dy/dx into the equation, and check that both sides match as expressions.
Read or sketch a slope field	Evaluate dy/dx at each point for the segment slope. A solution curve stays tangent to the segments it passes through.
Find the general solution	Separate variables, integrate both sides, and keep exactly one arbitrary constant.
Find the particular solution	Find the general solution first, then substitute the initial condition to solve for the constant.
Exponential model question	Recognize $dy/dt = ky$, write $y = Ce^{kt}$, and let C be the starting value $y(0)$.

PAIR	KEEP THEM STRAIGHT
General solution vs particular solution	General has the arbitrary constant and describes the whole family. Particular has the constant pinned down by an initial condition and describes one curve.
Verifying vs solving	Verifying starts with a candidate function and checks it. Solving starts with the equation and produces the function.
Horizontal segment vs $y = 0$	A horizontal segment means $dy/dx = 0$ at that point. It says nothing about the value of y there.
Growth ($k > 0$) vs decay ($k < 0$)	The sign of k in $dy/dt = ky$ decides. Positive k grows, negative k decays, no matter what the story problem is about.
kt in the exponent vs k alone	The solution is $y = Ce^{kt}$. At time t the exponent is the product kt , so $y(10)$ for $k = 0.2$ uses e^2 .
Dividing by $h(y)$ vs the lost solution	Separation divides by $h(y)$, which assumes $h(y) \neq 0$. The constant function $y = 0$ can be a solution the separated form misses.
$\ln y $ vs a power rule slip	$\int (1/y) dy = \ln y $. Treating it like $\int y dy$ is the integration error that ruins the whole separation.
Initial condition in the DE vs in the solution	The ordered pair goes into the general solution to find C , never into the differential equation itself.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Which of the following is a differential equation?

- A. $y = 3x^2 - 1$
- B. $dy/dx = 3x^2 - 1$
- C. $y(2) = 11$
- D. $\int 3x^2 dx$

2. Which function is a solution of $dy/dx = 4y$?

- A. $y = 4e^x$
- B. $y = e^x + 4$
- C. $y = 4x$
- D. $y = e^{4x}$

3. A slope field is drawn for the differential equation $dy/dx = y$. Which statement about the segments is true?

- A. All segments are horizontal.
- B. Segments along the line $y = 0$ are horizontal, and segments get steeper farther from the x -axis.
- C. Segments along the line $x = 0$ are horizontal.
- D. All segments have slope 1.

4. What is the general solution of $dy/dx = 3x^2/y$?

- A. $y^2 = x^3 + C$
- B. $y = 2x^3 + C$
- C. $y^2 = 2x^3 + C$
- D. $y^2 = 2x^3$

5. Find the particular solution of $dy/dx = 2x + 1$ with the initial condition $y(0) = 5$.

- A. $y = x^2 + x + 5$
- B. $y = x^2 + x + 4$
- C. $y = x^2 + 5$
- D. $y = 2x + 5$

6. The function y satisfies $dy/dx = -2xy$ and $y(1) = 3$. Which is the particular solution?

- A. $y = 3e^{-x^2}$
- B. $y = 3e^{1-x^2}$
- C. $y = 3e^{x^2}$
- D. $y = e^{1-x^2}$

7. A quantity satisfies $dy/dt = 0.2y$ with $y(0) = 50$. Which expression gives $y(10)$?

- A. $50e^2$
- B. $50e^{0.2}$
- C. $50 + 0.2(10)$
- D. $50e^{-2}$

8. A radioactive sample satisfies $dy/dt = -0.04y$, where y is grams and t is years, with $y(0) = 120$. After how many years will 60 grams remain?

- A. $t = 0.04/\ln 2$
- B. $t = 60/0.04$
- C. $t = \ln(0.04)/2$
- D. $t = (\ln 2)/0.04$

Answer Key

- 1. B.** A differential equation relates a function to its derivatives, and only B contains dy/dx . A is an ordinary function, C is a single value of a function, and D is an integral expression with no equation relating y to its derivative.
- 2. D.** For $y = e^{4x}$, $dy/dx = 4e^{4x} = 4y$, so both sides match. A puts the 4 on the wrong part: $dy/dx = 4e^x$ while $4y = 16e^x$. B adds a constant that survives on the right but not in the derivative. C gives $dy/dx = 4$ while $4y = 16x$.
- 3. B.** Since $dy/dx = y$, the slope at each point equals its y -coordinate. Where $y = 0$ the slope is 0, so those segments are horizontal, and $|\text{slope}|$ grows as $|y|$ grows. A describes $dy/dx = 0$, C swaps the roles of x and y , and D describes $dy/dx = 1$.
- 4. C.** Separating gives $y \, dy = 3x^2 \, dx$, so $y^2/2 = x^3 + C$, and multiplying through gives $y^2 = 2x^3 + C$ (the 2 absorbs into the constant). A forgets the $1/2$ from integrating $y \, dy$, B never integrates the y side at all, and D drops the constant, which loses the whole family.
- 5. A.** Integrating gives the general solution $y = x^2 + x + C$, and $y(0) = C = 5$ pins the constant down. B subtracts 1 from the constant for no reason, C fails to integrate the 1 term, and D never integrates in the first place.
- 6. B.** Separating gives $(1/y) \, dy = -2x \, dx$, so $\ln|y| = -x^2 + C$ and $y = Ae^{-x^2}$. The condition $y(1) = 3$ gives $3 = Ae^{-1}$, so $A = 3e$ and $y = 3e^{1-x^2}$. A uses the condition at $x = 0$ instead of $x = 1$, C flips the sign during separation, and D drops the factor of 3 from the initial value.
- 7. A.** The model gives $y = 50e^{0.2t}$, so at $t = 10$ the exponent is $0.2 \cdot 10 = 2$. B evaluates at $t = 1$, C treats the growth as linear by adding a fixed amount, and D flips the sign of k even though the model describes growth.
- 8. D.** From $y = 120e^{-0.04t}$, setting $y = 60$ gives $1/2 = e^{-0.04t}$, so $-\ln 2 = -0.04t$ and $t = (\ln 2)/0.04$. A inverts the fraction, B divides grams by a rate as if the decay were linear, and C takes the log of the rate constant instead of solving the equation.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Differential Equations deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Differential Equations deck and let spaced review bring them back over the next few days.

- State what a differential equation is, using $dy/dx = ky$ as your example.
- Explain what it means for a function to be a solution of a differential equation, and describe the substitution check.
- Show that $y = e^{-3x}$ satisfies $dy/dx = -3y$, showing each step.
- Describe how to draw one segment of a slope field for $dy/dx = x + y$ at the point $(1, 2)$.
- Explain what a solution curve does as it moves through a slope field.
- State where the segments of the slope field for $dy/dx = y - 2$ are horizontal, and why.
- Carry out separation of variables on $dy/dx = xy^2$, naming the assumption you make when you divide.
- Explain why the general solution has exactly one arbitrary constant.
- Given $dy/dx = \cos x$ with $y(0) = 1$, find the particular solution and check the initial condition.
- Explain the order of work: general solution first, then the initial condition.
- Write the exponential model differential equation and its solution, and state what the sign of k tells you.
- A rumor spreads with $dy/dt = 0.1y$ and $y(0) = 20$. Write $y(t)$ and explain what each number means.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Differential Equations deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Differential equation, Solution of a differential equation, Slope field, Separation of variables, General solution, Particular solution, Initial condition, Exponential model.

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