

Unit 6: Integration and Accumulation of Change

Unit 6 builds integration from the ground up. It starts with accumulation of change, approximates area with Riemann sums, defines the definite integral as a limit of those sums, connects differentiation and integration through the Fundamental Theorem of Calculus, and finishes with the basic techniques for finding antiderivatives. CED topics 6.1 through 6.6 and 6.8 through 6.10.

AP Calculus AB

Integration

About 12 minutes to read

How to use this guide

Read it in order the first time because the ideas stack. Accumulation gives the motivation, Riemann sums give the approximation, the definite integral makes it exact, the Fundamental Theorem connects it to everything you learned about derivatives, and the last topics give you the tools to compute antiderivatives by hand. Exam questions move back and forth across that whole chain, so keep the connections in view.

After the first read, use the trap boxes and the tables for the distinctions the exam tests most. The left versus right versus midpoint question and the two directions of the Fundamental Theorem show up constantly. Finish with the practice questions, then complete the recall check on the last page out loud.

What this unit is worth. Unit 6 carries 10 to 15 percent of the AP Calculus AB exam, and its ideas run through Units 7 and 8 as well. Every differential equation and every area or volume application later in the course depends on the integral defined here.

6.1 Exploring Accumulations of Change

Accumulation of change is the net total change of a quantity over an interval, found by accumulating its rate of change. If you know how fast something is changing at each moment, adding up all those little changes gives you the total change from start to finish.

The classic example is velocity and distance. If $v(t)$ is a car's velocity in miles per hour, the distance it travels between $t = a$ and $t = b$ is the accumulation of v over that interval. The same idea works anywhere. If water flows into a tank at $r(t)$ gallons per minute, the total water added between two times is the accumulation of the rate r . Units 1 through 5 were about rates. This unit is about recovering totals from rates.

Trap. Accumulation gives net change, not the final value. If a tank already holds 40 gallons and 15 more flow in, the accumulation is 15 and the final amount is 55. Questions that ask for the total accumulated are not asking for the ending amount.

6.2 Approximating Areas with Riemann Sums

A **Riemann sum** approximates the area under a curve by adding up the areas of rectangles. Split $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$. On each subinterval, build a rectangle whose height is the function value at one chosen point. The sum is $\sum f(x_i^*) \cdot \Delta x$ over all subintervals.

Which point you choose gives the four named types. A **left Riemann sum** uses the function value at the left endpoint of each subinterval. A **right Riemann sum** uses the right endpoint. A **midpoint Riemann sum** uses the midpoint. A **trapezoidal sum** uses trapezoids instead of rectangles, so each subinterval contributes $(\Delta x/2) \cdot [f(x_{i-1}) + f(x_i)]$.

Watch one example all the way through. For $f(x) = x^2$ on $[0, 2]$ with $n = 2$, the width is $\Delta x = 1$. The left sum uses $f(0)$ and $f(1)$, giving $0 + 1 = 1$. The right sum uses $f(1)$ and $f(2)$, giving $1 + 4 = 5$. The midpoint sum uses $f(0.5)$ and $f(1.5)$, giving $0.25 + 2.25 = 2.5$. The exact area turns out to be $8/3$, about 2.67, so the midpoint was closest here.

SUM TYPE	WHEN F IS INCREASING	WHEN F IS DECREASING
Left	Underestimates, because each rectangle sits below the rising curve	Overestimates, because each rectangle sits above the falling curve
Right	Overestimates, because each rectangle sits above the rising curve	Underestimates, because each rectangle sits below the falling curve
Midpoint	Usually closer to exact than left or right with the same n	Usually closer to exact than left or right with the same n
Trapezoidal	Equals the average of the left and right sums	Equals the average of the left and right sums

Trap. Left does not always underestimate. Whether a left sum is too big or too small depends on whether the function is rising or falling on each subinterval. For an increasing function, left underestimates and right overestimates. For a decreasing function it is the reverse. Questions love to flip the direction.

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

Summation (sigma) notation writes sums compactly. $\sum_{i=1}^n a_i$ means $a_1 + a_2 + \dots + a_n$. The i is the index, 1 is where it starts, and n is where it stops. So $\sum_{i=1}^4 i^2 = 1 + 4 + 9 + 16 = 30$. Every Riemann sum is a sigma sum, and the exam expects you to read and write them.

The **definite integral** is the exact signed area under f from a to b , defined as the limit of Riemann sums as $n \rightarrow \infty$. In symbols, $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum f(x_i^*) \cdot \Delta x$. As the rectangles get narrower, the approximation error shrinks to zero and the sum becomes the exact area.

Trap. A definite integral is signed area, not total area. Area above the x -axis counts positive and area below counts negative. If a region dips below the axis, the integral subtracts that part. A question that asks for total area wants you to split the integral at the x -intercepts, not just integrate straight across.

6.4 The Fundamental Theorem of Calculus and Accumulation Functions

The **Fundamental Theorem of Calculus** says differentiation and integration are inverse processes, and it comes in two directions that the exam tests separately.

First direction. If $F(x) = \int_a^x f(t) dt$ for a continuous f , then $F'(x) = f(x)$. Differentiating an accumulation function just returns the original function. So if $F(x) = \int_2^x t^3 dt$, then $F'(x) = x^3$. You do not integrate and then differentiate. The theorem skips the middle step.

Second direction. $\int_a^b f(x) dx = F(b) - F(a)$, where F is any antiderivative of f . To evaluate a definite integral, find an antiderivative and subtract its value at the bottom limit from its value at the top limit. For example, $\int_1^2 3x^2 dx = [x^3]_1^2 = 8 - 1 = 7$.

Trap. The two directions answer different questions. The first direction differentiates an integral with a variable upper limit. The second evaluates an integral with constant limits using an antiderivative. When you see d/dx in front of an integral, use the first. When you see numbers on both limits, use the second.

6.5 Interpreting the Behavior of Accumulation Functions Involving Area

An **accumulation function** is defined by $A(x) = \int_a^x f(t) dt$. It gives the accumulated net area under f from a up to x . Its graph is a running total of signed area, so its behavior comes straight from the sign of f .

Where f is positive, area is being added and A is increasing. Where f is negative, area is being subtracted and A is decreasing. Where f crosses from positive to negative, A reaches a relative maximum. Where f crosses from negative to positive, A reaches a relative minimum. And $A(a) = 0$ always, because integrating from a to a accumulates nothing.

Work through the logic once. Suppose f is positive on $(0, 2)$, zero at $x = 2$, and negative on $(2, 4)$, and $A(x) = \int_0^x f(t) dt$. Then A increases on $(0, 2)$ and decreases on $(2, 4)$, so A has its maximum value at $x = 2$. You never need the formula for f . The sign chart of f is the sign chart of A' .

Trap. Zeros of f are not zeros of A . A zero of f is where accumulation stops growing, which is a high or low point of A , not a place where A equals zero. The only automatic zero of A is at the lower limit, $A(a) = 0$.

6.6 Applying Properties of Definite Integrals

The **properties of definite integrals** let you split, combine, and flip integrals without computing them.

PROPERTY	WHAT IT SAYS
Sum and difference	$\int_a^b [f \pm g] dx = \int_a^b f dx \pm \int_a^b g dx$
Constant multiple	$\int_a^b c \cdot f dx = c \cdot \int_a^b f dx$
Reversed limits	$\int_a^b f dx = -\int_b^a f dx$
Zero-width interval	$\int_a^a f dx = 0$
Additivity	$\int_a^b f dx + \int_b^c f dx = \int_a^c f dx$

These turn data into answers. If $\int_1^4 f(x) dx = 6$ and $\int_4^9 f(x) dx = 2$, then $\int_1^9 f(x) dx = 6 + 2 = 8$ by additivity. If $\int_0^2 f(x) dx = 3$, then $\int_2^0 f(x) dx = -3$ by reversed limits.

Trap. Additivity only joins intervals that touch end to end. $\int_1^3 + \int_3^7 = \int_1^7$ works, but $\int_1^3 + \int_4^7$ does not combine because of the gap. Also, flipping the limits flips the sign. $\int_3^1 f$ is the negative of $\int_1^3 f$, not the same thing.

6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

An **antiderivative** of f is a function F with $F'(x) = f(x)$. Any two antiderivatives of the same function differ by a constant, because the derivative of a constant is zero.

The **indefinite integral** is the whole family of antiderivatives, written $\int f(x) dx = F(x) + C$. The $+C$ is the constant of integration, and it is part of the answer. Leaving it off makes an indefinite integral wrong.

The workhorse is the power rule in reverse. For $n \neq -1$, $\int x^n dx = x^{n+1}/(n+1) + C$. Raise the exponent by one, then divide by the new exponent. So $\int (4x^3 - 2x) dx = 4 \cdot (x^4/4) - 2 \cdot (x^2/2) + C = x^4 - x^2 + C$. You can always check by differentiating. The derivative of $x^4 - x^2 + C$ is $4x^3 - 2x$, which matches.

Trap. $+C$ belongs on indefinite integrals and never on definite ones. A definite integral evaluates to a number through $F(b) - F(a)$, where the constant cancels anyway. Writing $+C$ after a definite integral is a notation error the exam will mark wrong.

6.9 Integrating Using Substitution

Integration by substitution is the Chain Rule in reverse. When the integrand contains a function and something close to its derivative, set $u = g(x)$, compute $du = g'(x) dx$, and rewrite the whole integral in terms of u .

Take $\int 2x(x^2 + 1)^3 dx$. The inner function is $x^2 + 1$ and its derivative $2x$ is sitting right there. Set $u = x^2 + 1$, so $du = 2x dx$. The integral becomes $\int u^3 du = u^4/4 + C = (x^2 + 1)^4/4 + C$. For a definite integral, either change the limits to u -values when you substitute or substitute back to x before evaluating. With $u = x^2 + 1$, $\int_0^1 2x(x^2 + 1)^3 dx$ becomes $\int_1^2 u^3 du = [u^4/4]_1^2 = (16 - 1)/4 = 15/4$.

Trap. Substitute the differential too, not just the u . The most common error is replacing $x^2 + 1$ with u but leaving a stray x or dx behind. Every x must be accounted for, either inside u or inside du . If a leftover x cannot be absorbed, that substitution was the wrong choice.

6.10 Integrating Functions Using Long Division and Completing the Square

Some rational integrands need to be rewritten before any rule applies. **Integrating with long division and completing the square** means reshaping the integrand into a form whose antiderivative is elementary.

When the numerator's degree is at least the denominator's, divide first. For $(x^3 + 1)/(x + 1)$, long division gives $x^2 - x + 1$ with no remainder, since $(x + 1)(x^2 - x + 1) = x^3 + 1$. Then $\int (x^3 + 1)/(x + 1) dx = \int (x^2 - x + 1) dx = x^3/3 - x^2/2 + x + C$.

Completing the square handles denominators that are quadratics with no real roots. For $\int dx/(x^2 + 6x + 10)$, rewrite the denominator as $(x + 3)^2 + 1$. With $u = x + 3$ and $du = dx$, the integral becomes $\int du/(u^2 + 1) = \arctan(u) + C = \arctan(x + 3) + C$.

Trap. Do not split a fraction by "canceling" across a sum. $(x^3 + 1)/(x + 1)$ is not $x^2 + 1$, and it is not $x^3/x + 1/1$. Division has to be done properly, term by term, before you integrate. If the degrees tell you the fraction is top-heavy, divide first and integrate the result.

Confusions That Cost Points

PAIR	KEEP THEM STRAIGHT
Accumulation vs. final value	Accumulation is the net change over the interval. Add the starting value separately if the question wants the ending amount.
FTC first direction vs. second direction	d/dx of $\int_a^x f(t) dt$ gives $f(x)$. A definite integral with constant limits gives $F(b) - F(a)$. Different question, different move.
Definite vs. indefinite integral	Definite has limits and evaluates to a number, no $+C$. Indefinite has no limits and needs $+C$.
Left vs. right Riemann sum	Check whether f is increasing or decreasing before deciding which one overestimates. Left is not automatically the low one.
Signed area vs. total area	The integral subtracts area below the x -axis. Total area requires splitting at the x -intercepts.
u vs. du in substitution	Replace the inner function with u and its derivative times dx with du . Nothing with x may remain.
Reversed limits	$\int_a^b f = -\int_b^a f$. Flipping the limits flips the sign.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. For $f(x) = x^2$ on the interval $[0, 2]$, the left Riemann sum with $n = 2$ subintervals is equal to
 - A. 2.5
 - B. 5
 - C. 1
 - D. $8/3$
2. If $F(x) = \int_2^x t^3 dt$, then $F'(x) =$
 - A. $3x^2$
 - B. x^3
 - C. $x^3 - 8$
 - D. t^3
3. $\int_1^2 3x^2 dx =$
 - A. 8
 - B. $6x$
 - C. 9
 - D. 7
4. Let f be continuous, with f positive on $(0, 2)$, $f(2) = 0$, and f negative on $(2, 4)$. If $A(x) = \int_0^x f(t) dt$, then A attains its maximum value at $x =$
 - A. 2
 - B. 0
 - C. 4
 - D. It cannot be determined from the given information
5. If $\int_1^3 f(x) dx = 5$ and $\int_3^7 f(x) dx = 8$, then $\int_1^7 f(x) dx =$
 - A. 3
 - B. 40
 - C. 13
 - D. -13

6. $\int (4x^3 - 2x) \, dx =$

A. $x^4 - x^2$

B. $x^4 - x^2 + C$

C. $12x^2 - 2 + C$

D. $4x^4 - 2x^2 + C$

7. $\int 2x(x^2 + 1)^3 \, dx =$

A. $(x^2 + 1)^4 + C$

B. $2x(x^2 + 1)^4/4 + C$

C. $4x(x^2 + 1)^3 + C$

D. $(x^2 + 1)^4/4 + C$

8. $\int (x^3 + 1)/(x + 1) \, dx =$

A. $x^3/3 + x + C$

B. $x^3/3 - x^2/2 + x + C$

C. $x^3/3 - x^2/2 + x$

D. $3x^2 + C$

Answer Key

- 1. C.** With $n = 2$ on $[0, 2]$, $\Delta x = 1$ and the left endpoints are 0 and 1, so the sum is $f(0) + f(1) = 0 + 1 = 1$. A is the midpoint sum, B is the right sum, and D is the exact value of the integral, which is what a Riemann sum approximates, not what it equals.
- 2. B.** By the first direction of the Fundamental Theorem, differentiating $\int_2^x t^3 dt$ returns the integrand with x in place of t , so x^3 . A differentiates the integrand instead of applying the theorem. C confuses this with evaluating a definite integral, which would subtract $F(2)$. D leaves the dummy variable t in the answer.
- 3. D.** An antiderivative of $3x^2$ is x^3 , so $[x^3]_1^2 = 8 - 1 = 7$. A evaluates only at the top limit and forgets to subtract $F(1)$. B differentiates instead of antidifferentiating. C adds $F(1)$ instead of subtracting it.
- 4. A.** $A'(x) = f(x)$, so A increases where f is positive and decreases where f is negative. The switch from positive to negative at $x = 2$ makes it a relative maximum of A. B is where A equals zero, not where it peaks. C is an endpoint where A has been decreasing. D is wrong because the sign change of f determines the behavior of A completely.
- 5. C.** The intervals $[1, 3]$ and $[3, 7]$ touch end to end, so additivity gives $5 + 8 = 13$. A subtracts the two values, which no property justifies. B multiplies them, which is never how integrals combine. D applies a sign flip as if the limits had been reversed, which they were not.
- 6. B.** By the reverse power rule, $\int 4x^3 dx = x^4$ and $\int 2x dx = x^2$, and the family of antiderivatives needs $+C$. A drops the constant of integration. C differentiates term by term instead of antidifferentiating. D raises the exponent but forgets to divide by the new exponent.
- 7. D.** With $u = x^2 + 1$, $du = 2x dx$, so the integral is $\int u^3 du = u^4/4 + C$. Differentiating the answer gives back $2x(x^2 + 1)^3$, which checks. A drops the factor of $1/4$ from the power rule. B substitutes u but leaves the $2x$ behind instead of absorbing it into du . C differentiates the integrand.
- 8. B.** Long division gives $(x^3 + 1)/(x + 1) = x^2 - x + 1$, and integrating term by term gives $x^3/3 - x^2/2 + x + C$. A "cancels" across the sum to get $x^2 + 1$, which is not equal to the original fraction. C does the work correctly but omits $+C$. D differentiates the numerator instead of dividing and integrating.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Integration deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Integration deck and let spaced review bring them back over the next few days.

- Explain accumulation of change in your own words, and give one example with units.
- Write the formula for a Riemann sum using sigma notation, and say what Δx equals.
- For an increasing function, say which of left and right Riemann sums overestimates and why.
- State the definite integral as a limit of Riemann sums.
- Explain why a definite integral is signed area rather than total area.
- State both directions of the Fundamental Theorem of Calculus and say when to use each.
- Given only the sign of f , explain how to find where an accumulation function A increases, decreases, and attains its maximum.
- State all five properties of definite integrals from memory.
- Explain why every indefinite integral needs $+C$ and every definite integral must not have it.
- Integrate $\int (6x^2 + 5) dx$ from scratch, showing the reverse power rule.
- Walk through a u -substitution: how to choose u , what du is, and what to do with the limits in a definite integral.
- Explain when polynomial long division is needed before integrating a rational function.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Integration deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Accumulation of change, Riemann sum, Left Riemann sum, Right Riemann sum, Midpoint Riemann sum, Trapezoidal sum, Definite integral, Summation (sigma) notation, Fundamental Theorem of Calculus, Accumulation function, Properties of definite integrals, Antiderivative, Indefinite integral, Integration by substitution, Integrating with long division and completing the square.

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