

Unit 5: Analytical Applications of Differentiation

Unit 5 turns the derivative into conclusions: where functions rise and fall, where they bend, where their highest and lowest values occur, and how to optimize a quantity. CED topics 5.1 through 5.7 and 5.10.

AP Calculus AB

Analytical Applications

About 12 minutes to read

How to use this guide

Read it in order the first time because the tools build on each other. The Mean Value Theorem and Extreme Value Theorem state what is guaranteed, critical points locate the candidates, the derivative tests classify them, and optimization puts the whole chain to work. Exam questions usually give a function or a graph and ask you to justify a conclusion, so pay attention to the conditions each theorem requires.

After the first read, use the trap boxes and the tables to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

Why this unit matters. This is the justification unit. Free-response questions here rarely accept a bare answer. They ask why a function has a local maximum at a point, or why an absolute maximum must exist on an interval, and the justification has to name the test or theorem and confirm its conditions were met. The conditions matter as much as the conclusions, so learn each hypothesis alongside the result it produces.

5.1 Using the Mean Value Theorem

The **Mean Value Theorem (MVT)** connects the average rate of change to the instantaneous rate. If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is some c in (a, b) with $f'(c) = [f(b) - f(a)] / (b - a)$. In plain terms, somewhere inside the interval the instantaneous rate of change equals the average rate of change over the whole interval.

Worked example. Let $f(x) = x^2$ on $[1, 4]$. The average rate of change is $[f(4) - f(1)] / (4 - 1) = (16 - 1) / 3 = 5$. Since $f'(x) = 2x$, setting $2c = 5$ gives $c = 2.5$, which lies in $(1, 4)$. So at $x = 2.5$ the tangent line is parallel to the secant line joining the endpoints.

Trap. The MVT is about the derivative, not the function value. If a question asks for a point where $f'(c)$ equals the average rate, that is the MVT. If it asks for a point where f takes some intermediate value, that is the Intermediate Value Theorem. Also note the split conditions: continuity is required on the closed interval $[a, b]$, but differentiability is only required on the open interval (a, b) . A function that is continuous but not differentiable at an interior point does not satisfy the hypotheses.

5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

The **Extreme Value Theorem (EVT)** says that if f is continuous on the closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$. It guarantees existence, not location. On an open interval there is no such guarantee: $f(x) = x$ on $(0, 1)$ approaches 0 and 1 but never attains either.

A **local (relative) extremum** is a function value $f(c)$ that is greater than (maximum) or less than (minimum) all nearby function values. An **absolute (global) extremum** is the largest or smallest value of the function over its entire domain or over a given interval. A local maximum is not automatically the absolute maximum. The absolute maximum is simply the biggest among the local maxima and the endpoint values.

A **critical point** is a value c in the domain of f where $f'(c) = 0$ or where $f'(c)$ does not exist. Every local extremum occurs at a critical point, which makes critical points the short list of suspects. But a critical point is not automatically an extremum. For $f(x) = x^3$, $f'(x) = 3x^2$, so $x = 0$ is a critical point, yet the function keeps increasing through it and has no local max or min there.

Worked example. For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. Setting $f'(x) = 0$ gives $x = 1$ and $x = -1$. Both are in the domain, so both are critical points. Whether either is an extremum is decided by the tests in 5.4 and 5.7, not by the critical point alone.

Trap. Both ways of being a critical point count. A point where the derivative equals zero is a critical point, and a point in the domain where the derivative does not exist is also a critical point. Questions that ask you to list all critical points are checking whether you remembered the second kind.

5.3 Determining Intervals on Which a Function is Increasing or Decreasing

A function is **increasing on an interval where $f'(x) > 0$** and **decreasing where $f'(x) < 0$** , for f differentiable there. The sign of the derivative is the whole story. Find the critical points, split the domain into intervals between them, and test the sign of f' on each interval.

Worked example. For $f(x) = x^2 - 4x + 3$, $f'(x) = 2x - 4$. The derivative is negative when $x < 2$ and positive when $x > 2$. So f is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

Trap. Increasing and decreasing are decided by the sign of f' , not the sign of f . A function can be positive and decreasing at the same time. When a question asks where f is increasing, test f' , and do not let a positive function value distract you.

5.4 Using the First Derivative Test to Determine Relative (Local) Extrema

The **First Derivative Test** classifies a critical point by watching the sign of f' change around it. If f' changes from positive to negative at a critical point c , f has a local maximum at c . If f' changes from negative to positive at c , f has a local minimum at c . If f' does not change sign at c , there is no local extremum there.

Worked example. Returning to $f(x) = x^2 - 4x + 3$ with $f'(x) = 2x - 4$: the only critical point is $x = 2$, and f' changes from negative to positive there. The First Derivative Test concludes f has a local minimum at $x = 2$, with value $f(2) = 4 - 8 + 3 = -1$.

Trap. A sign change is required, not just a zero of the derivative. For $g(x) = x^4$, $g'(x) = 4x^3$ is zero at $x = 0$ but negative on both sides, so there is no local extremum at $x = 0$ from this test alone. Never classify a critical point without checking the sign on both sides.

5.5 Using the Candidates Test to Determine Absolute (Global) Extrema

The **Candidates Test** finds absolute extrema of a continuous function on a closed interval $[a, b]$. Evaluate f at every critical point in (a, b) and at the endpoints a and b . The largest of those values is the absolute maximum and the smallest is the absolute minimum. The EVT guarantees the extrema exist, and this test locates them.

Worked example. For $f(x) = x^3 - 3x^2$ on $[-1, 3]$: $f'(x) = 3x^2 - 6x = 3x(x - 2)$, so the critical points are $x = 0$ and $x = 2$. Evaluate: $f(-1) = -1 - 3 = -4$, $f(0) = 0$, $f(2) = 8 - 12 = -4$, $f(3) = 27 - 27 = 0$. The absolute maximum is 0, attained at $x = 0$ and $x = 3$. The absolute minimum is -4 , attained at $x = -1$ and $x = 2$.

Trap. The Candidates Test is for absolute extrema, and the endpoints must be included. A frequent error is evaluating only the critical points and missing an absolute extremum that occurs at an endpoint. Also, the test needs a closed interval and a continuous function. On an open interval, the largest candidate value might not be attained at all.

5.6 Determining Concavity of Functions over Their Domains

Concavity describes how a graph bends. A function is concave up on an interval where $f''(x) > 0$, shaped like a cup, and concave down where $f''(x) < 0$, shaped like a cap. A **point of inflection** is a point where the concavity changes and the tangent line exists there.

Worked example. For $f(x) = x^3$, $f''(x) = 6x$. The second derivative is negative when $x < 0$ and positive when $x > 0$, so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. The concavity changes at $(0, 0)$, which is a point of inflection.

Trap. Concavity comes from f'' , not from f or f' . A function can be increasing and concave down at the same time. And a point of inflection is not a critical point: critical points come from f' , inflection points come from a sign change in f'' . Keep the two derivatives assigned to their own jobs.

5.7 Using the Second Derivative Test to Determine Extrema

The **Second Derivative Test** classifies a critical point c where $f'(c) = 0$ using concavity. If $f''(c) > 0$, the graph is concave up there and f has a local minimum at c . If $f''(c) < 0$, the graph is concave down there and f has a local maximum at c . If $f''(c) = 0$, the test is inconclusive and you must fall back to the First Derivative Test.

Worked example. For $f(x) = x^3 - 3x^2 + 1$: $f'(x) = 3x^2 - 6x = 3x(x - 2)$, so the critical points are $x = 0$ and $x = 2$. With $f''(x) = 6x - 6$: $f''(0) = -6 < 0$, so f has a local maximum at $x = 0$. $f''(2) = 6 > 0$, so f has a local minimum at $x = 2$.

Trap. The inconclusive case is real and tested. For $h(x) = x^4$, $h'(0) = 0$ and $h''(0) = 0$, so the Second Derivative Test says nothing, yet $x = 0$ is a local minimum. When $f''(c) = 0$, switch to the First Derivative Test instead of guessing.

5.10 Introduction to Optimization Problems

Optimization means finding the maximum or minimum value of a quantity. The method: write the quantity as a function of one variable, use the given constraint to eliminate the other variables, then use derivatives to locate extrema. Because optimization asks for the best value, always verify that the critical point you found is a maximum or a minimum, using the First or Second Derivative Test or by checking the endpoints of the domain.

Worked example. A farmer has 100 feet of fence to enclose a rectangular pen against a straight barn, so only three sides need fencing. Let x be the depth of the pen and $100 - 2x$ the length along the barn.

Area $A(x) = x(100 - 2x) = 100x - 2x^2$, with $0 < x < 50$. Then $A'(x) = 100 - 4x = 0$ gives $x = 25$, so the length is $100 - 50 = 50$. Since $A''(x) = -4 < 0$, this critical point is a maximum. The pen should be 25 feet deep and 50 feet long, for a maximum area of 1250 square feet.

Trap. Finding a critical point is only half the job. Optimization questions ask for the maximum or the minimum, so confirm which one you have. If the domain has endpoints, evaluate there too, because the optimum of a continuous function on a closed interval can sit at an endpoint rather than at a critical point.

Which tool answers which question

QUESTION BEING ASKED	TOOL TO USE
Where is f increasing or decreasing?	Sign of f' on intervals between critical points
Does f have a local max or min at c ?	First Derivative Test (sign change of f') or Second Derivative Test (sign of $f''(c)$)
What are the absolute max and min on $[a, b]$?	Candidates Test: critical points plus endpoints
Where does the graph bend, and where does the bend change?	Sign of f'' ; points of inflection where it changes sign
Must some point have a given instantaneous rate?	Mean Value Theorem, after checking continuity and differentiability
Must an absolute max exist at all?	Extreme Value Theorem, for f continuous on a closed interval

Confusions That Cost Points

Most missed questions in this unit come from a short list of pairs that look alike under time pressure. Review each pair carefully so you can tell them apart when you see them in a question.

PAIR	HOW TO SEPARATE THEM
Mean Value Theorem vs Intermediate Value Theorem	MVT gives a point where the derivative equals the average rate. IVT gives a point where the function takes an intermediate value. Derivative versus value.
Critical point vs local extremum	A critical point is where f' is zero or undefined. A local extremum is a nearby high or low. Every local extremum is at a critical point, but a critical point like $x = 0$ for x^3 may be neither.
Local vs absolute extremum	Local compares to nearby values. Absolute compares to every value on the interval or domain. The absolute max is the largest of the local maxima and the endpoint values.
First Derivative Test vs Candidates Test	The First Derivative Test classifies one critical point as a local max, local min, or neither. The Candidates Test compares critical points and endpoints to find absolute extrema on a closed interval.
First vs Second Derivative Test	The First uses the sign change of f' around the point. The Second uses the sign of $f''(c)$ at the point. The Second is quicker but goes silent when $f''(c) = 0$.
Concave up vs increasing	Increasing is about f' being positive. Concave up is about f'' being positive. A graph can rise while bending downward.
Point of inflection vs critical point	Inflection comes from a sign change in f'' . Critical points come from f' being zero or undefined. They are found with different derivatives.
$f'(c) = 0$ vs $f'(c)$ undefined	Both make c a critical point. When listing critical points, include the points where the derivative fails to exist, not only where it equals zero.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

- 1.** Let f be continuous on $[1, 5]$ and differentiable on $(1, 5)$, with $f(1) = 2$ and $f(5) = 10$. Which of the following is guaranteed by the Mean Value Theorem?
- A. There exists c in $(1, 5)$ such that $f'(c) = 2$
 - B. There exists c in $(1, 5)$ such that $f(c) = 2$
 - C. $f'(x) = 2$ for all x in $(1, 5)$
 - D. The Mean Value Theorem does not apply because the hypotheses do not guarantee differentiability at $x = 1$ and $x = 5$
- 2.** Consider $f(x) = x^3$ on the closed interval $[-2, 2]$. Which of the following statements is true?
- A. f has a local maximum at $x = 0$
 - B. The absolute maximum of f on $[-2, 2]$ is 8
 - C. The absolute minimum of f on $[-2, 2]$ is 0
 - D. The Extreme Value Theorem does not apply because f is not differentiable at $x = 0$
- 3.** Let $f(x) = x^3 - 3x^2 - 9x + 1$. On which interval is f decreasing?
- A. $(-\infty, -1)$
 - B. $(-1, 3)$
 - C. $(3, \infty)$
 - D. $(-1, \infty)$
- 4.** Let $g(x) = x^4 - 4x^3$. Which of the following is true about the critical points of g ?
- A. g has a local maximum at $x = 0$
 - B. g has a local minimum at $x = 3$
 - C. g has a local maximum at $x = 3$
 - D. g has local extrema at both $x = 0$ and $x = 3$

5. What is the absolute maximum of $f(x) = x^3 - 3x$ on the closed interval $[-2, 2]$?

- A. 2
- B. -2
- C. 8
- D. The Candidates Test cannot be used because f is not continuous on $[-2, 2]$

6. Let $f(x) = x^3 + 3x^2$. On which interval is f concave down?

- A. $(-1, \infty)$
- B. $(-\infty, -1)$
- C. $(-\infty, -2)$
- D. $(0, \infty)$

7. Let $f(x) = x^3 - 3x^2$. The Second Derivative Test concludes that $x = 2$ is

- A. a local maximum
- B. a local minimum
- C. neither, because $f''(2) = 0$
- D. a point of inflection

8. A rectangle has perimeter 40. What dimensions maximize its area?

- A. 10 by 10
- B. 20 by 20
- C. 5 by 15
- D. 8 by 12

Answer Key

1. A. The average rate of change is $[f(5) - f(1)] / (5 - 1) = (10 - 2) / 4 = 2$, so the MVT guarantees some c in $(1, 5)$ with $f'(c) = 2$. B is the IVT trap: it asks about a function value instead of the derivative, and 2 is attained only at the endpoint $x = 1$ for a strictly increasing function. C overclaims: the MVT gives one point, not every point. D misstates the hypotheses: differentiability is required only on the open interval $(1, 5)$, not at the endpoints.

2. B. The candidates are the critical point $x = 0$ ($f'(x) = 3x^2 = 0$) and the endpoints: $f(-2) = -8$, $f(0) = 0$, $f(2) = 8$. The largest is 8, so the absolute maximum is 8 at $x = 2$. A is the critical-point trap: f' does not change sign at $x = 0$, so there is no local extremum there. C confuses the critical-point value with the minimum; the absolute minimum is -8 . D is false twice over: f is differentiable at $x = 0$ ($f'(0) = 0$ exists), and the EVT requires only continuity anyway.

3. B. $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$, which is negative between its zeros $x = -1$ and $x = 3$. Since f is decreasing where $f' < 0$, the answer is $(-1, 3)$. A and C name the intervals where $f' > 0$, where f is increasing. D is the partial trap: it starts at the right critical point but runs past $x = 3$, where the sign flips.

4. B. $g'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$, so the critical points are $x = 0$ and $x = 3$. The sign of g' is negative on both sides of $x = 0$ (no sign change, so no extremum there) and changes from negative to positive at $x = 3$, giving a local minimum. A and D fall for the critical-point trap at $x = 0$, where the derivative is zero but the sign never changes. C reverses the sign reading at $x = 3$.

5. A. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, so the critical points in $(-2, 2)$ are $x = -1$ and $x = 1$. Evaluating all candidates: $f(-2) = -8 + 6 = -2$, $f(-1) = -1 + 3 = 2$, $f(1) = 1 - 3 = -2$, $f(2) = 8 - 6 = 2$. The largest value is 2. B names the absolute minimum, not the maximum. C evaluates x^3 at $x = 2$ but drops the $-3x$ term. D is false: a polynomial is continuous everywhere, so the Candidates Test applies.

6. B. $f''(x) = 6x + 6$, which is negative when $x < -1$, so f is concave down on $(-\infty, -1)$. A reverses the inequality and names the concave-up interval. C and D mistake critical points of f ($x = -2$ and $x = 0$, where $f' = 0$) for the inflection point; concavity comes from f'' , whose zero is at $x = -1$.

7. B. $f'(x) = 3x^2 - 6x = 3x(x - 2)$, so $x = 2$ is a critical point. $f''(x) = 6x - 6$, and $f''(2) = 6 > 0$, so the graph is concave up there and $x = 2$ is a local minimum. A reverses the test. C miscomputes: $f''(2) = 6$, not 0. D confuses the test: the Second Derivative Test classifies extrema; a point of inflection requires a sign change in f'' .

8. A. Let the width be w , so the length is $20 - w$ and area $A(w) = w(20 - w) = 20w - w^2$. Then $A'(w) = 20 - 2w = 0$ gives $w = 10$, and $A''(w) = -2 < 0$ confirms a maximum. The rectangle is 10 by 10 with area 100. B has the wrong perimeter (80, not 40). C and D both have perimeter 40 but smaller areas (75 and 96), which is the point of the optimization: satisfying the constraint is necessary but not sufficient.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Analytical Applications deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Analytical Applications deck and let spaced review bring them back over the next few days.

- State the Mean Value Theorem with both of its hypotheses, and explain what each hypothesis rules out.
- Explain the difference between the Mean Value Theorem and the Intermediate Value Theorem in one sentence each.
- State the Extreme Value Theorem, and give an example of a function on an open interval that attains no maximum.
- Define a critical point, and name both ways a point qualifies.
- Explain why $x = 0$ is a critical point of $f(x) = x^3$ but not a local extremum.
- Describe how to find the intervals where a function is increasing or decreasing, and what sign you are testing.
- State the First Derivative Test, including the case where it concludes neither.
- Walk through the Candidates Test for $f(x) = x^3 - 3x^2$ on $[-1, 3]$ without notes, including the endpoints.
- Define concave up and concave down using f'' , and define a point of inflection.
- State the Second Derivative Test, and describe what to do when $f''(c) = 0$.
- For $h(x) = x^4$, explain why the Second Derivative Test is inconclusive at $x = 0$ and what the First Derivative Test says instead.
- Outline the steps of an optimization problem, and explain why you must verify max versus min at the end.
- Set up (but do not solve) an optimization problem: a box with a square base and volume 32 has to minimize surface area. Name the variable, the constraint, and the function to differentiate.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Analytical Applications deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Mean Value Theorem (MVT), Extreme Value Theorem (EVT), Local (relative) extremum, Absolute (global) extremum, Critical point, Increasing and decreasing intervals, First Derivative Test, Candidates Test, Concavity, Point of inflection, Second Derivative Test, Optimization.

About this guide. Written for Rycal and aligned to the College Board AP Calculus AB course framework, Unit 5. All questions and explanations are original Rycal writing. Rycal is independent and is not affiliated with or endorsed by the College Board.