

Unit 4: Contextual Applications of Differentiation

Unit 4 puts derivatives to work on motion, changing quantities, approximations, and hard limits. It covers position, velocity, and acceleration, related rates, linearization, and L'Hospital's Rule.

AP Calculus AB

Applications of Differentiation

About 12 minutes to read

How to use this guide

Read the four topics in order. Straight-line motion is the foundation for interpreting any rate, related rates applies implicit differentiation to changing quantities, linearization is the first practical use of the tangent line, and L'Hospital's Rule finishes the limit techniques the exam expects. Each topic has one or two worked examples. Work them yourself before reading the solution.

After the first read, use the trap boxes and the comparison tables to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Unit 4 is one of the most heavily tested units on the AP Calculus AB exam, typically 10 to 15 percent of the multiple-choice section. Motion questions appear almost every year, and related rates shows up as a free-response favorite. The techniques here also return in later units, especially when optimization and accumulation build on the same ideas.

4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration

A **position function** $s(t)$ gives an object's location at time t along a line. Everything in this topic follows from differentiating it. The **velocity** is the instantaneous rate of change of position, $v(t) = s'(t)$. Its sign tells you the direction of motion. Positive velocity means the object is moving right or up, and negative velocity means it is moving left or down. The **acceleration** is the instantaneous rate of change of velocity, $a(t) = v'(t) = s''(t)$. Its sign tells you whether velocity is increasing or decreasing, not which way the object is moving.

The **speed** is the magnitude of velocity, $|v(t)|$. Speed is never negative. When $v(t) = 0$, the object is at rest at that instant, which means its speed is zero too, but the sign of velocity alone never tells you the speed.

The motion chain

QUANTITY	WHAT IT IS	WHAT ITS SIGN MEANS
Position $s(t)$	Location at time t	Positive means right of the origin, negative means left
Velocity $v(t) = s'(t)$	Rate of change of position	Positive means moving right, negative means moving left, zero means at rest
Acceleration $a(t) = v'(t)$	Rate of change of velocity	Positive means velocity increasing, negative means velocity decreasing
Speed = $ v(t) $	Magnitude of velocity	Always zero or positive. Never negative

Trap. Velocity and speed are not the same. If $v(3) = -4$, the speed at $t = 3$ is 4, and the object is moving left. A question asking for speed wants the absolute value, and a question asking for direction wants the sign.

When an object speeds up or slows down

An object **speeds up** when velocity and acceleration have the same sign, because the velocity is growing in magnitude. It **slows down** when they have opposite signs. This is a sign comparison, and it is the single most tested idea in motion problems.

Worked example

Suppose $s(t) = t^2 - 6t + 8$ for t at least 0. Then $v(t) = 2t - 6$ and $a(t) = 2$. The particle is at rest when $v(t) = 0$, so $2t - 6 = 0$ gives $t = 3$. For $t < 3$, v is negative and a is positive, so they have opposite signs and the particle is slowing down. For $t > 3$, v is positive and a is positive, so they have the same sign and the particle is speeding up. The position at $t = 3$ is $s(3) = 9 - 18 + 8 = -1$.

4.4 Introduction to Related Rates

Related rates problems involve several quantities that all change with time and are linked by an equation. The method is to differentiate that equation implicitly with respect to time, then substitute the known values to find the unknown rate.

The order of operations matters. Differentiate first, using the chain rule to attach d/dt factors to every changing quantity. Only after differentiating do you plug in the values known at the instant described. Substituting a value for a changing quantity before differentiating freezes it as a constant and breaks the chain rule.

The related-rates method

1. Write the equation that relates the changing quantities.
2. Differentiate both sides with respect to t , attaching d/dt factors with the chain rule.
3. Substitute the known rates and the values at the given instant.
4. Solve for the unknown rate, and include units.

Worked example

A circle expands so that its radius increases at 3 cm/s. How fast is the area increasing when the radius is 5 cm? Start with $A = \pi r^2$. Differentiate with respect to t : $dA/dt = 2\pi r(dr/dt)$. Now substitute $r = 5$ and $dr/dt = 3$: $dA/dt = 2\pi(5)(3) = 30\pi \text{ cm}^2/\text{s}$. Notice that $r = 5$ was not substituted until after differentiating. If you had written $A = 25\pi$ first, differentiating would give $dA/dt = 0$, which is wrong.

Trap. Never substitute the instant's values before differentiating. The radius being 5 cm at one moment does not make it constant. Differentiate with r still a variable, then substitute.

4.6 Approximating Values of a Function Using Local Linearity and Linearization

Local linearity is the property that a differentiable function looks like a straight line when you zoom in close enough at a point. The **linearization**, also called the tangent line approximation, turns that idea into a formula: $f(x)$ is approximately $f(a) + f'(a)(x - a)$ for x near a , using the tangent line at a .

This is the same tangent line you already know how to write. The only new idea is using it as an approximation machine. Pick a point a where f and f' are easy to compute, then feed in x values near a .

Worked example

Approximate $\sqrt{26}$ using linearization at $a = 25$. Let $f(x) = \sqrt{x}$, so $f(25) = 5$ and $f'(x) = 1/(2\sqrt{x})$, which gives $f'(25) = 1/10$. The linearization is $L(x) = 5 + (1/10)(x - 25)$. At $x = 26$: $L(26) = 5 + (1/10)(1) = 5.1$. The actual value is about 5.099, so the approximation is off by about 0.001.

Trap. The $(x - a)$ factor uses the distance from a , not x itself. With $a = 25$ and $x = 26$, the factor is 1, not 26. Writing $5 + (1/10)(26)$ is the most common linearization error.

Linearization is only reliable near the base point. The farther x moves from a , the worse the approximation gets, because the function curves away from its tangent line. Exam questions always place x close to a when they expect this method.

4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

An **indeterminate form** is a limit form like $0/0$ or ∞/∞ whose value cannot be determined from the form alone. It requires further analysis. A form like $0/2$ is not indeterminate. It equals 0 directly, because only the numerator is going to zero.

L'Hospital's Rule says that if the limit of $f(x)/g(x)$ has indeterminate form $0/0$ or ∞/∞ , then the limit equals the limit of $f'(x)/g'(x)$, provided that limit exists. You differentiate the numerator and the denominator separately. You do not use the quotient rule.

Worked examples

For the limit as $x \rightarrow 0$ of $(e^x - 1)/x$: substituting gives $0/0$, which is indeterminate. Differentiating top and bottom separately gives $e^x/1$, and as $x \rightarrow 0$ that approaches 1. So the limit is 1.

For the limit as $x \rightarrow \infty$ of $(\ln x)/x$: substituting gives ∞/∞ , which is indeterminate. Differentiating top and bottom gives $(1/x)/1 = 1/x$, and as $x \rightarrow \infty$ that approaches 0. So the limit is 0.

Trap. L'Hospital's Rule applies only to indeterminate forms. If substitution gives $0/2$ or $3/0$, the rule does not apply, and differentiating top and bottom gives a wrong answer. Always check the form first.

Confusions That Cost Points

Most missed questions in this unit come from a short list of pairs that look alike under time pressure. Review each pair carefully so you can tell them apart when you see them in a question.

PAIR	HOW TO SEPARATE THEM
Velocity vs speed	Velocity has a sign that gives direction. Speed is the absolute value and is never negative.
At rest vs speeding up	At rest means $v = 0$ at one instant. Speeding up means v and a share a sign over an interval.
Acceleration sign vs direction	Positive acceleration means velocity is increasing, which can happen while moving left. It does not mean moving right.
Differentiate vs substitute first	In related rates, differentiate with respect to t first, then substitute the instant's values. Substituting first freezes a changing quantity as a constant.
$(x - a)$ vs x in linearization	The tangent line approximation uses the distance from the base point a . Using x alone multiplies by the wrong amount.
$0/0$ vs $0/2$	$0/0$ is indeterminate and allows L'Hospital's Rule. $0/2$ is just 0 , and the rule does not apply.
L'Hospital's Rule vs quotient rule	L'Hospital's Rule differentiates top and bottom separately. The quotient rule combines them. Using the quotient rule on a limit is always wrong.
Indeterminate vs undefined	Indeterminate means more analysis is needed. A form like $3/0$ is not indeterminate; it grows without bound or fails to exist.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. A particle moves along the x-axis with position $s(t) = t^3 - 6t^2 + 9t$ for $0 \leq t \leq 5$. On which interval is the particle moving to the left?
 - A. $(0, 1)$
 - B. $(1, 3)$
 - C. $(3, 5)$
 - D. $(0, 3)$
2. The velocity of a particle is $v(t) = 2t - 8$. What is the speed of the particle at $t = 5$?
 - A. -2
 - B. 2
 - C. 10
 - D. 0
3. A particle moves with $s(t) = t^2 - 4t$. For which values of t is the particle speeding up?
 - A. $t < 2$
 - B. $t > 2$
 - C. $t = 2$
 - D. for all t
4. A spherical balloon is inflated so that its volume increases at $10 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 2 cm?
 - A. $5/(8\pi) \text{ cm/s}$
 - B. 10 cm/s
 - C. $5/(2\pi) \text{ cm/s}$
 - D. $5\pi/8 \text{ cm/s}$

5. A 13-foot ladder slides down a wall. The bottom of the ladder moves away from the wall at 2 ft/s. How fast is the top of the ladder sliding down when the bottom is 5 ft from the wall?
- A. $-5/6$ ft/s
 - B. $5/6$ ft/s
 - C. $-5/12$ ft/s
 - D. $-6/5$ ft/s
6. Let f be differentiable with $f(4) = 3$ and $f'(4) = 1/4$. Use linearization at $a = 4$ to approximate $f(4.2)$.
- A. 3.05
 - B. 4.05
 - C. 3.2
 - D. 3.005
7. What is the limit as $x \rightarrow 0$ of $\sin(3x)/x$?
- A. 0
 - B. 3
 - C. 1
 - D. $1/3$
8. What is the limit as $x \rightarrow 1$ of $(x^2 - 1)/(x + 1)$?
- A. 0
 - B. 2
 - C. $1/2$
 - D. The limit does not exist

Answer Key

- 1. B.** $v(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$. Velocity is negative on $(1, 3)$, which means the particle moves left there. A and C are intervals where v is positive, so the particle moves right. D includes $(0, 1)$, where the particle moves right, so it overclaims.
- 2. B.** $v(5) = 2(5) - 8 = 2$, and speed is $|v|$, so 2. A reports the velocity without taking the absolute value, which is the speed-versus-velocity trap. C drops the -8 when computing $v(5)$, reading $2(5)$ alone. D confuses the zero of velocity, which occurs at $t = 4$, with the requested time.
- 3. B.** $v(t) = 2t - 4$ and $a(t) = 2$. The particle speeds up when v and a share a sign. Since a is always positive, that happens when v is positive, which is $t > 2$. A is the interval where v is negative while a is positive, so the particle is slowing down there. C is the instant the particle is at rest, not an interval of speeding up. D ignores that the sign of v changes.
- 4. A.** $V = (4/3)\pi r^3$, so $dV/dt = 4\pi r^2(dr/dt)$. Substituting $dV/dt = 10$ and $r = 2$ gives $10 = 16\pi(dr/dt)$, so $dr/dt = 10/(16\pi) = 5/(8\pi)$ cm/s. B reads the given rate as the answer without relating the quantities. C comes from a power-rule error that drops the r^2 factor. D inverts the division.
- 5. A.** With $x^2 + y^2 = 169$, differentiating gives $2x(dx/dt) + 2y(dy/dt) = 0$. At $x = 5$, $y = 12$ and $dx/dt = 2$, so $20 + 24(dy/dt) = 0$ and $dy/dt = -5/6$ ft/s. The negative sign means the top is sliding down. B drops that sign. C loses a factor of 2 in the implicit differentiation. D inverts the fraction.
- 6. A.** $L(x) = 3 + (1/4)(x - 4)$, so $L(4.2) = 3 + (1/4)(0.2) = 3.05$. B uses x instead of $(x - a)$, computing $3 + (1/4)(4.2)$. C misreads $f'(4)$ as 1. D slips a decimal place, computing $(1/4)(0.02)$.
- 7. B.** The form is $0/0$, so L'Hospital's Rule applies: differentiating top and bottom gives $3\cos(3x)/1$, which approaches 3 as $x \rightarrow 0$. A assumes $0/0$ equals 0. C applies the standard $\sin x/x$ limit while ignoring the factor of 3. D inverts the answer.
- 8. A.** Substituting $x = 1$ gives $0/2 = 0$, which is not indeterminate, so the limit is 0 directly. B is the trap of misapplying L'Hospital's Rule to a non-indeterminate form, since $2x/1$ approaches 2. C is an arithmetic slip. D assumes the fraction is undefined because the numerator is zero, but the denominator is 2, not 0.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Contextual Applications deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Contextual Applications deck and let spaced review bring them back over the next few days.

- State how position, velocity, and acceleration are connected by differentiation.
- Explain the difference between velocity and speed, and give the speed when $v = -6$.
- State the sign rule for speeding up versus slowing down, and apply it to $v(t) = t - 5$ with $a(t) = 1$.
- Explain why acceleration being positive does not mean the object is moving right.
- List the four steps of the related-rates method in order.
- Explain why you must differentiate before substituting the instant's values.
- Write the linearization formula and explain what $(x - a)$ represents.
- Approximate $\sqrt{50}$ using linearization at $a = 49$, showing each step.
- State L'Hospital's Rule and the two forms it applies to.
- Explain why $0/2$ is not indeterminate and what its limit is.
- Describe what goes wrong if you apply L'Hospital's Rule to $(x^2 - 1)/(x + 1)$ as $x \rightarrow 1$.
- Give one example each of a $0/0$ limit and an ∞/∞ limit, and evaluate both.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Contextual Applications deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Position function, velocity, acceleration, speed, related rates, local linearity, linearization (tangent line approximation), L'Hospital's Rule, indeterminate form.

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