

Unit 3: Differentiation: Composite, Implicit, and Inverse Functions

Unit 3 extends differentiation to composite functions with the Chain Rule, to equations that define y implicitly, to inverse functions including $\arcsin$, $\arctan$, and $\arccos$, and to second and higher-order derivatives. CED topics 3.1, 3.2, 3.3, 3.4, and 3.6.

AP Calculus AB

Differentiation

About 12 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. The Chain Rule is the engine for everything else in this unit. Implicit differentiation is the Chain Rule applied to an equation instead of a function. Inverse function and inverse trig derivatives both come from the same reciprocal relationship. Higher-order derivatives reuse the rules you already know.

After the first read, use the trap boxes and the tables to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

Why this unit matters. Almost every function on the AP exam that is worth differentiating is a composite function. The Chain Rule shows up in related rates, in every inverse and implicit problem, and in any derivative of e^u , $\ln(u)$, or a trig function of something complicated. Students who can name the outside and inside of a composition on sight move through the rest of the course much faster.

3.1 The Chain Rule

A **composite function** applies one function to the output of another. In $f(g(x))$, g is the inside function and f is the outside function. In $\sin(x^2)$, the inside is x^2 and the outside is sine. In $(4x - 1)^5$, the inside is $4x - 1$ and the outside is the fifth power. Naming both parts before you differentiate is the whole skill.

The **Chain Rule** says the derivative of $f(g(x))$ is $f'(g(x)) \cdot g'(x)$. In words, take the derivative of the outside, keep the inside exactly as it is, then multiply by the derivative of the inside. Both multiplications are required. The most common error in this unit is stopping after the first half.

Worked example. Let $y = (4x - 1)^5$. The outside is the fifth power, so its derivative form is $5(\text{inside})^4$. The inside is $4x - 1$, whose derivative is 4. Multiply: $y' = 5(4x - 1)^4 \cdot 4 = 20(4x - 1)^4$.

Some composites hide more than two layers. In $e^{\sin(3x)}$, the outermost layer is the exponential, the middle is sine, and the innermost is $3x$. Differentiate from the outside in: $e^{\sin(3x)} \cdot \cos(3x) \cdot 3 = 3e^{\sin(3x)}\cos(3x)$. Every layer contributes one factor.

Outside and inside pairs worth memorizing

COMPOSITE	DERIVATIVE
u^n	$nu^{n-1} \cdot u'$
$\sin(u), \cos(u)$	$\cos(u) \cdot u', -\sin(u) \cdot u'$
e^u	$e^u \cdot u'$
$\ln(u)$	u'/u
$\sqrt{u}$	$u' / (2\sqrt{u})$

Trap. The derivative of $(3x + 2)^4$ is not $4(3x + 2)^3$. That answer differentiated the outside and forgot the inside. The full answer is $4(3x + 2)^3 \cdot 3 = 12(3x + 2)^3$. When a question looks like a one-step power rule, check whether something sits inside the parentheses first.

Trap. Do not substitute the inner derivative into the outside function. For $y = \sin(x^3)$, the answer is $3x^2\cos(x^3)$, not $3x^2\cos(3x^2)$. The inside stays untouched inside the cosine. The inner derivative multiplies the whole result.

3.2 Implicit Differentiation

Some equations define y as a function of x without solving for y . The circle $x^2 + y^2 = 25$ is the standard example. There is no single $y = \dots$ formula, but y still depends on x , so y still has a derivative.

Implicit differentiation means differentiating both sides of the equation while treating y as a function of x . Every y -term gets the Chain Rule treatment, which produces a dy/dx factor. Then solve for dy/dx .

Worked example. Differentiate $x^2 + y^2 = 25$. The derivative of x^2 is $2x$. The derivative of y^2 is $2y \cdot dy/dx$, because y is the inside function and its derivative with respect to x is dy/dx . The derivative of 25 is 0. So $2x + 2y(dy/dx) = 0$, and solving gives $dy/dx = -x/y$.

The product rule and quotient rule combine with implicit differentiation when the equation mixes x and y in one term. For $xy = 6$, differentiating the left side needs the product rule: $1 \cdot y + x \cdot dy/dx = 0$, so $dy/dx = -y/x$.

Trap. The derivative of y^2 with respect to x is $2y(dy/dx)$, not $2y$. Treating y like a constant drops the chain rule factor and breaks every later step. Any term containing y must carry a dy/dx when you differentiate the equation.

Trap. When the algebra gives $dy/dx = -x/y$, the negative sign is part of the answer. Moving $2x$ across the equals sign flips its sign. Students who write x/y usually lost the negative during the rearrangement, not during the differentiation.

3.3 Differentiating Inverse Functions

If f is invertible and $f(a) = b$, then the **derivative of an inverse function** follows the rule $(f^{-1})'(b) = 1 / f'(a)$, as long as $f'(a)$ is not zero. The derivative of the inverse at the y -value b is the reciprocal of the derivative of the original at the matching x -value a .

The point-swapping is the point. You are given information about f at a , and the question asks about f^{-1} at b , where $b = f(a)$. Worked example: f is invertible, $f(2) = 5$, and $f'(2) = 4$. Then $(f^{-1})'(5) = 1 / f'(2) = 1/4$. The 5 tells you which point of f to use, and the 2 tells you where to evaluate f' .

Trap. $(f^{-1})'(b)$ uses $f'(a)$, evaluated at a , not at b . Writing $1 / f'(b)$ mixes up which point belongs to which function. Read the given pair $f(a) = b$ as directions: b is the input to the inverse, a is where you evaluate f' .

3.4 Differentiating Inverse Trigonometric Functions

The AB course requires three inverse trig derivatives. They are worth memorizing exactly, because the exam tests them directly and they also appear inside chain rule problems.

FUNCTION	DERIVATIVE
$\arcsin x$	$1 / \sqrt{1 - x^2}$
$\arccos x$	$-1 / \sqrt{1 - x^2}$
$\arctan x$	$1 / (1 + x^2)$

These almost always arrive with a chain rule attached. For $y = \arcsin(3x)$, the derivative is $1/\sqrt{1 - (3x)^2}$ times the derivative of $3x$: $3/\sqrt{1 - 9x^2}$. For $y = \arctan(x^2)$, it is $1/(1 + (x^2)^2)$ times $2x$: $2x/(1 + x^4)$.

Trap. $\arcsin$ and $\arccos$ differ only by a sign. The derivative of $\arcsin x$ is $1/\sqrt{1 - x^2}$. The derivative of $\arccos x$ is the same fraction with a negative sign. Questions that swap the two are testing whether you attached the minus to the right function.

Trap. $\arctan$ gets the $1/(1 + x^2)$ form, no square root. Mixing it with the $\arcsin$ form produces $1/\sqrt{1 + x^2}$, which matches nothing in the course. Keep the root with $\arcsin$ and $\arccos$ only.

3.6 Calculating Higher-Order Derivatives

The **second derivative** is the derivative of the derivative, written $f''(x)$, y'' , or d^2y/dx^2 . It measures how the rate of change itself is changing. **Higher-order derivatives** continue the pattern: $f'''(x)$ or $f^{(3)}(x)$ for the third, $f^{(4)}(x)$ for the fourth, and $f^{(n)}(x)$ in general.

Computing them is just repeated differentiation with the rules from Units 2 and 3. Worked example: $f(x) = x^4 - 3x^2 + 2x$. Then $f'(x) = 4x^3 - 6x + 2$, and differentiating again gives $f''(x) = 12x^2 - 6$. Each pass lowers every exponent by one. A third pass would give $f^{(3)}(x) = 24x$.

Chain rule composites survive into higher orders too. For $y = (2x + 1)^3$, the first derivative is $6(2x + 1)^2$ and the second is $24(2x + 1)$. The inner derivative factor of 2 appears in every pass, so it accumulates.

Trap. $f''(x)$ and $(f(x))^2$ are different objects. The first is the second derivative. The second is the first function squared. Notation questions exploit this, so read whether the exponent-like mark sits on the f or on the parentheses.

Confusions That Cost Points

PAIR	KEEP THEM STRAIGHT
$(f(g(x)))'$ vs $f'(g(x))$	The first is the full chain rule: $f'(g(x)) \cdot g'(x)$. The second is only the outside differentiated, with the inside left in place. The exam choice that drops $g'(x)$ is the trap.
dy/dx vs dx/dy	Implicit differentiation solves for dy/dx . If the algebra is inverted, you have dx/dy instead. For $x^2 + y^2 = 25$, $dy/dx = -x/y$ but $dx/dy = -y/x$.
$(f^{-1})'(b)$ vs $1/f'(b)$	The reciprocal uses $f'(a)$, evaluated at the x -value a where $f(a) = b$. Evaluating at b is the standard wrong answer.
$d/dx[\arcsin x]$ vs $d/dx[\arccos x]$	Same denominator $\sqrt{1 - x^2}$. Only $\arccos$ carries the negative sign.
Chain rule vs product rule	Use the chain rule when one function sits inside another, like $(x^2 + 1)^3$. Use the product rule when two functions multiply side by side, like $x^2 \sin(x)$. The structure of the expression decides, not how complicated it looks.

Practice Questions

1. If $y = (2x - 5)^3$, then $dy/dx =$

- A. $3(2x - 5)^2$
- B. $6(2x - 5)^2$
- C. $6(2x - 5)^3$
- D. $3(2x - 5)^2 + 2$

2. If $y = \sin(x^3)$, then $dy/dx =$

- A. $\cos(x^3)$
- B. $3\cos(x^3)$
- C. $3x^2\cos(x^3)$
- D. $3x^2\cos(3x^2)$

3. For the curve $x^2 + y^2 = 25$, $dy/dx =$

- A. x/y
- B. $-y/x$
- C. 0
- D. $-x/y$

4. For the curve $x^3 + y^3 = 9$, $dy/dx =$

- A. $-x^2/y^2$
- B. x^2/y^2
- C. $-x/y$
- D. $-y^2/x^2$

Practice Questions

5. Let f be an invertible function with $f(3) = 7$ and $f'(3) = 5$. What is $(f^{-1})'(7)$?

- A. 5
- B. $1/7$
- C. $1/5$
- D. $-1/5$

6. $d/dx [\arcsin(3x)] =$

- A. $1/\sqrt{1 - 9x^2}$
- B. $3/\sqrt{1 - 9x^2}$
- C. $-3/\sqrt{1 - 9x^2}$
- D. $3/(1 + 9x^2)$

7. $d/dx [\arccos x] =$

- A. $1/\sqrt{1 - x^2}$
- B. $1/(1 + x^2)$
- C. $-1/(1 + x^2)$
- D. $-1/\sqrt{1 - x^2}$

8. If $f(x) = x^4 - 3x^2 + 2x$, then $f''(x) =$

- A. $12x^2 - 6$
- B. $4x^3 - 6x + 2$
- C. $12x^2 - 6x + 2$
- D. $24x$

Answer Key

- 1. B.** $6(2x - 5)^2$. The outside derivative is $3(2x - 5)^2$ and the inside derivative is 2, so multiply to get $6(2x - 5)^2$. A drops the inner derivative entirely, the classic chain rule trap. C lowers nothing: it keeps the exponent 3 instead of reducing it to 2. D treats the composition as a sum and adds the inner derivative instead of multiplying.
- 2. C.** $3x^2\cos(x^3)$. Differentiate the outside sine to cosine, leave x^3 untouched inside it, and multiply by the inner derivative $3x^2$. A forgot the inner derivative. B wrote the inner derivative as 3, dropping the x^2 factor. D substituted the inner derivative into the cosine instead of multiplying by it, so the argument wrongly became $3x^2$.
- 3. D.** $-x/y$. Differentiating gives $2x + 2y(dy/dx) = 0$, so $dy/dx = -2x/(2y) = -x/y$. A lost the negative sign when rearranging, the most common algebra slip in this problem. B gives $-y/x$, which is dx/dy , the reciprocal of the asked-for derivative. C treats the constant 25 as if it forces the whole derivative to zero, confusing the derivative of a constant with the derivative of an equation.
- 4. A.** $-x^2/y^2$. Differentiating gives $3x^2 + 3y^2(dy/dx) = 0$, so $dy/dx = -3x^2/(3y^2) = -x^2/y^2$. B lost the negative sign in the rearrangement. C "canceled" a power from numerator and denominator that was never a common factor. D inverted the fraction, giving the reciprocal of the correct derivative.
- 5. C.** $1/5$. The inverse at 7 uses the point where f equals 7, which is $x = 3$, so $(f^{-1})'(7) = 1/f'(3) = 1/5$. A skipped the reciprocal and reported $f'(3)$ directly. B evaluated the reciprocal at the wrong input, using 7 where the definition requires the x -value 3. D attached a negative sign that the reciprocal rule never produces.
- 6. B.** $3/\sqrt{1 - 9x^2}$. The arcsin derivative is $1/\sqrt{1 - u^2}$ with $u = 3x$, multiplied by $u' = 3$. A applied the arcsin formula but forgot the chain rule factor of 3. C used the arccos sign on an arcsin problem. D grabbed the arctan formula $1/(1 + u^2)$ instead of the arcsin one.
- 7. D.** $-1/\sqrt{1 - x^2}$. arccos is the one inverse trig derivative in this course with a negative sign. A gives the arcsin derivative, missing the minus. B gives the arctan derivative, which has no square root at all. C mixes both errors, pairing the arccos sign with the arctan form.
- 8. A.** $12x^2 - 6$. The first derivative is $4x^3 - 6x + 2$, and differentiating once more gives $12x^2 - 6$. B stopped after the first derivative. C differentiated $4x^3$ correctly but only partially differentiated the rest, leaving $-6x + 2$ instead of -6 . D differentiated a third time and reported $f^{(3)}(x)$ instead of the asked-for second derivative.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed a chain rule question, the same forgotten-inner-derivative trap is worth practicing again in Rycal, where the Differentiation deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud. Check the ones you cannot explain yet and turn them into flashcards.

- State the Chain Rule in words and in symbols for $f(g(x))$.
- Identify the inside and outside functions in $\ln(\cos(2x))$.
- Find $d/dx[(5x + 1)^4]$ without looking at the worked example.
- Explain why $e^{\sin(3x)}$ needs three multiplied factors in its derivative.
- Describe the two steps of implicit differentiation.
- Find dy/dx for $x^2 + y^2 = 25$ from scratch.
- Explain what goes wrong if you write $d/dx(y^2) = 2y$.
- State the inverse function derivative rule and explain the point-swapping.
- Given $f(2) = 5$ and $f'(2) = 4$, find $(f^{-1})'(5)$.
- Write the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$ from memory.
- Explain which of the three inverse trig derivatives carries a negative sign and why students mix it up.
- Find $d/dx[\arctan(x^2)]$, showing the chain rule step.
- Explain the difference between $f''(x)$ and $(f(x))^2$.
- Find the second derivative of $y = (2x + 1)^3$.
- Name the two trap answers for $y = (2x - 5)^3$ and what error each one represents.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Differentiation deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Chain Rule, Composite function, Implicit differentiation, Derivative of an inverse function, Derivative of $\arcsin x$, Derivative of $\arctan x$, Derivative of $\arccos x$, Second derivative, Higher-order derivatives.

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