

Unit 2: Differentiation: Definition and Fundamental Properties

Unit 2 builds the derivative from average and instantaneous rates of change, then develops the rules that compute it: the power rule, the constant, sum, difference, and constant multiple rules, the derivatives of the basic transcendental functions, and the product and quotient rules.

AP Calculus AB

Differentiation

About 12 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. The derivative starts as a rate of change at a point, notation gives you several ways to write it, the connection between differentiability and continuity tells you where derivatives can fail, and then the rules let you compute derivatives quickly without the limit definition. Exam questions usually give a function and ask for its derivative, or they test whether you know when a rule applies and which one.

After the first read, use the trap boxes and the tables to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

Why this unit matters. Every topic after this one depends on these rules. The chain rule, implicit differentiation, related rates, curve sketching, and eventually integration all assume you can differentiate the basic functions and combine them with the product and quotient rules without hesitation. Fluency here makes every later unit faster.

2.1 Defining Average and Instantaneous Rates of Change at a Point

A **secant line** passes through two points on a curve, and its slope is the **average rate of change** of the function between those points. For $f(x) = x^2$, the average rate from $x = 1$ to $x = 3$ is $(9 - 1) / (3 - 1) = 4$. A **tangent line** passes through $(c, f(c))$ with slope $f'(c)$, and that slope is the **instantaneous rate of change** of f at c . When the two points on a secant line slide closer together, the secant slope approaches the tangent slope. The average rate becomes the instantaneous rate.

Trap. The secant slope is an average over an interval. The tangent slope is a rate at one point. A question that gives two points and asks for the average rate of change wants the secant computation, not $f'(c)$.

2.2 Defining the Derivative of a Function and Using Derivative Notation

The **derivative of a function** at x is the instantaneous rate of change there, defined by $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)] / h$, provided the limit exists. This limit says to compute the average rate over smaller and smaller intervals around x and see what it approaches. **Derivative notation** comes in several forms that all mean the same thing: $f'(x)$ and y' are prime notation, dy/dx is Leibniz notation, and $d/dx[f(x)]$ names the operator. For example, if $f(x) = 3x^2$, then $f'(x) = \lim_{h \rightarrow 0} [3(x+h)^2 - 3x^2] / h = \lim_{h \rightarrow 0} (6x + 3h) = 6x$.

Trap. dy/dx is a single symbol for the derivative, not a fraction you can split apart in Unit 2. Treat d/dx as an instruction that means differentiate what follows with respect to x .

2.3 Estimating Derivatives of a Function at a Point

The **difference quotient** $[f(a+h) - f(a)] / h$ approximates $f'(a)$, and smaller values of h give better estimates. If you have a table of values, pick the entries closest to the point of interest and compute the quotient. For $f(x) = x^3$, estimating $f'(2)$ with $h = 0.1$ gives $(2.1^3 - 2^3) / 0.1 = (9.261 - 8) / 0.1 = 12.61$, which is close to the exact value $3(2)^2 = 12$.

Trap. The difference quotient with h still in it is an estimate, not the derivative. The derivative is the limit as $h \rightarrow 0$. Do not report $6 + h$ as $f'(3)$ for $f(x) = x^2$. The limit of $6 + h$ as $h \rightarrow 0$ is 6.

2.4 Connecting Differentiability and Continuity

A function is **differentiable** at $x = c$ when $f'(c)$ exists, which means the limit defining the derivative exists there. **Differentiability implies continuity:** if f is differentiable at c , then f is continuous at c . The direction matters because the converse is false. A function can be continuous at a point and still not be differentiable there. The three cases the course names are a **sharp corner** (the absolute value function at $x = 0$), a **cusp**, and a **vertical tangent**. At each one the graph has no single tangent slope.

Trap. Differentiable implies continuous, but continuous does not imply differentiable. A corner is the classic counterexample. When a question asks whether differentiability follows from continuity, the answer is no.

2.5 Applying the Power Rule

The **Power Rule** says that if $f(x) = x^n$, then $f'(x) = n \cdot x^{n-1}$. Multiply by the exponent, then subtract one from it. Before applying it, rewrite roots and fractions as powers. For $f(x) = x^5$, $f'(x) = 5x^4$. For $g(x) = \sqrt{x} = x^{1/2}$, $g'(x) = (1/2)x^{-1/2} = 1/(2\sqrt{x})$. For $h(x) = 1/x^2 = x^{-2}$, $h'(x) = -2x^{-3} = -2/x^3$.

Trap. Rewrite first, then differentiate. Applying the power rule directly to $1/x^2$ without rewriting is where sign and exponent errors start.

2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple

Four rules let you differentiate term by term. The **Constant Rule** says the derivative of a constant is zero: $d/dx[c] = 0$. The **Sum Rule** says $d/dx[f + g] = f' + g'$. The **Difference Rule** says $d/dx[f - g] = f' - g'$. The **Constant Multiple Rule** says constants factor out: $d/dx[c \cdot f(x)] = c \cdot f'(x)$. Applied together, if $f(x) = 4x^3 - 2x + 7$, then $f'(x) = 12x^2 - 2$, because the constant multiple rule pulls the 4 and the -2 out, the power rule handles the powers, and the constant 7 differentiates to zero.

RULE	FORMULA
Constant	$d/dx[c] = 0$
Constant multiple	$d/dx[c \cdot f(x)] = c \cdot f'(x)$
Sum	$d/dx[f + g] = f' + g'$
Difference	$d/dx[f - g] = f' - g'$

Trap. The derivative of a lone constant is zero, not the constant itself. In $f(x) = 4x^3 - 2x + 7$, the 7 disappears and the $-2x$ becomes -2 . A common miss is writing $+7$ or $-2x$ in the answer.

2.7 Derivatives of $\cos x$, $\sin x$, e^x , and $\ln x$

Four functions have derivatives worth memorizing exactly. The **derivative of $\sin x$** is $\cos x$. The **derivative of $\cos x$** is $-\sin x$. The **derivative of e^x** is e^x , the one function that is its own derivative. The **derivative of $\ln x$** is $1/x$, for $x > 0$.

FUNCTION	DERIVATIVE	WATCH FOR
$\sin x$	$\cos x$	Positive. The pair starts here
$\cos x$	$-\sin x$	The minus sign. Easy to drop
e^x	e^x	Unchanged. Not $x \cdot e^{x-1}$
$\ln x$	$1/x$	Valid for $x > 0$

These combine with the earlier rules. If $f(x) = 2\sin x - e^x + \ln x$, then $f'(x) = 2\cos x - e^x + 1/x$, using the constant multiple, difference, and sum rules on the three terms.

Trap. The power rule does not apply to e^x . Writing $x \cdot e^{x-1}$ treats e as a variable base, but e is a constant base, so the rule is $d/dx[e^x] = e^x$. Questions plant this distractor regularly.

2.8 The Product Rule

The **Product Rule** says $d/dx[f \cdot g] = f' \cdot g + f \cdot g'$. Differentiate the first factor times the second, plus the first times the derivative of the second. The derivative of a product is not the product of the derivatives. For $h(x) = x^2 \cdot \sin x$, $h'(x) = 2x \cdot \sin x + x^2 \cdot \cos x$.

Trap. The product rule has two terms joined by a plus sign. Writing only $f' \cdot g$ or only $f \cdot g'$ loses half the answer, and writing $f' \cdot g'$ invents a rule that does not exist.

2.9 The Quotient Rule

The **Quotient Rule** says $d/dx[f/g] = (f' \cdot g - f \cdot g') / g^2$. The numerator is the derivative of the top times the bottom, minus the top times the derivative of the bottom, and the denominator is the bottom squared. For $h(x) = (x^2 + 1) / (x - 1)$, the numerator becomes $2x(x - 1) - (x^2 + 1)(1) = x^2 - 2x - 1$, so $h'(x) = (x^2 - 2x - 1) / (x - 1)^2$.

Trap. The order in the numerator is fixed: $f' \cdot g$ comes first, then minus $f \cdot g'$. Reversing the subtraction or forgetting to square the denominator are the two most common errors, and both appear as answer choices.

2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

The remaining four trig derivatives are the **derivative of tan x**, $\sec^2 x$; the **derivative of cot x**, $-\csc^2 x$; the **derivative of sec x**, $\sec x \cdot \tan x$; and the **derivative of csc x**, $-\csc x \cdot \cot x$. Each follows from the quotient rule. For $\tan x = \sin x / \cos x$, the quotient rule gives $(\cos x \cdot \cos x - \sin x \cdot (-\sin x)) / \cos^2 x = (\cos^2 x + \sin^2 x) / \cos^2 x = 1 / \cos^2 x = \sec^2 x$.

FUNCTION	DERIVATIVE
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\csc x$	$-\csc x \cdot \cot x$

Trap. The co-functions carry the minus signs. Cos, cot, and csc all have negative derivatives. If your answer for $d/dx[\csc x]$ comes out positive, check the sign.

Confusions That Cost Points

PAIR	THE DISTINCTION
Secant slope vs tangent slope	Secant slope is an average rate over two points. Tangent slope is the instantaneous rate at one point, equal to $f'(c)$
Differentiable vs continuous	Differentiable at c implies continuous at c . The reverse fails at corners, cusps, and vertical tangents
Difference quotient vs derivative	The quotient $[f(a+h) - f(a)] / h$ estimates $f'(a)$. The derivative is the limit as $h \rightarrow 0$
Power rule on x^n vs on e^x	The power rule applies to variable bases. $d/dx[e^x] = e^x$, never $x \cdot e^{x-1}$
Derivative of a constant vs of cx	$d/dx[c] = 0$, but $d/dx[cx] = c$. A constant alone vanishes. A constant times x keeps the constant
Product rule vs product of derivatives	$d/dx[f \cdot g] = f' \cdot g + f \cdot g'$, two terms. $f' \cdot g'$ alone is wrong
Quotient numerator order	$(f' \cdot g - f \cdot g') / g^2$. The $f' \cdot g$ term comes first, and the denominator is squared
sin/cos sign vs tan/sec pattern	$d/dx[\cos x] = -\sin x$. The co-functions cos, cot, csc all take minus signs
$\ln x$ vs e^x	$d/dx[\ln x] = 1/x$ for $x > 0$. $d/dx[e^x] = e^x$. They are inverses, not the same rule

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. Let $f(x) = x^3$. What is the average rate of change of f on the interval $[1, 2]$?
 - A. 3
 - B. 7
 - C. 8
 - D. 12

2. For $f(x) = x^2$, the difference quotient $[f(3+h) - f(3)] / h$ simplifies to
 - A. $6 + h$
 - B. 6
 - C. $6 + 2h$
 - D. h

3. If f is a differentiable function, which of the following is NOT a standard notation for its derivative?
 - A. $f'(x)$
 - B. dy/dx
 - C. $d/dx[f(x)]$
 - D. $\Delta f / \Delta x$

4. Which of the following statements is always true?
 - A. If f is continuous at c , then f is differentiable at c
 - B. If f is differentiable at c , then f is continuous at c
 - C. If f has a sharp corner at c , then f is differentiable at c
 - D. If f is not differentiable at c , then f is not continuous at c

5. If $f(x) = 3/x + \sqrt{x}$, then $f'(x) =$

- A. $-3/x^2 + 1/(2\sqrt{x})$
- B. $3/x^2 + 1/(2\sqrt{x})$
- C. $-3/x^2 + \sqrt{x}/2$
- D. $-3/x^2 - 1/(2\sqrt{x})$

6. If $h(x) = x \cdot e^x$, then $h'(x) =$

- A. $e^x(1 + x)$
- B. $x \cdot e^x$
- C. $e^x + x^2 \cdot e^{x-1}$
- D. $1 + e^x$

7. If $g(x) = (\sin x) / x$, then $g'(x) =$

- A. $(x \cos x - \sin x) / x^2$
- B. $(\sin x - x \cos x) / x^2$
- C. $(\cos x) / 1$
- D. $(x \cos x - \sin x) / x$

8. $d/dx[\csc x] =$

- A. $-\csc x \cdot \cot x$
- B. $\csc x \cdot \cot x$
- C. $\sec x \cdot \tan x$
- D. $-\csc^2 x$

Answer Key

1. **B.** Average rate of change is the secant slope: $(f(2) - f(1)) / (2 - 1) = (8 - 1) / 1 = 7$. A is $f'(1)$, the instantaneous rate at the left endpoint, not the average. C is $f(2)$ alone, which ignores the formula entirely. D is $f'(2)$, the instantaneous rate at the right endpoint.
2. **A.** $[(3+h)^2 - 9] / h = (6h + h^2) / h = 6 + h$. B is the limit of the quotient as $h \rightarrow 0$, which is the derivative, not the quotient itself. C doubles the h term by mistake. D drops the $6h$ term from the expansion.
3. **D.** $\Delta f / \Delta x$ denotes a difference ratio, not the derivative. A is prime notation, B is Leibniz notation, and C applies the d/dx operator, and all three are standard for f' .
4. **B.** Differentiability at c implies continuity at c . A is the false converse: a corner like $|x|$ at 0 is continuous but not differentiable. C contradicts the corner case directly. D inverts the implication: a function can fail to be differentiable at a point where it is continuous.
5. **A.** Rewrite as $3x^{-1} + x^{1/2}$, then apply the power rule: $-3x^{-2} + (1/2)x^{-1/2} = -3/x^2 + 1/(2\sqrt{x})$. B flips the sign on the first term. C failed to subtract one from the exponent on $\sqrt{x}$. D flipped the sign on the second term.
6. **A.** Product rule: $1 \cdot e^x + x \cdot e^x = e^x(1 + x)$. B kept only the second half of the rule. C applied the power rule to e^x , giving $x^2 \cdot e^{x-1}$, which is the classic e^x error. D added the pieces instead of multiplying, treating a product like a sum.
7. **A.** Quotient rule: $(\cos x \cdot x - \sin x \cdot 1) / x^2$. B reversed the subtraction in the numerator. C differentiated the top and bottom separately and divided, which is not a rule. D forgot to square the denominator.
8. **A.** $d/dx[\csc x] = -\csc x \cdot \cot x$. B dropped the minus sign that all co-functions carry. C is the derivative of $\sec x$, not $\csc x$. D is the derivative of $\cot x$.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Differentiation deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Differentiation deck and let spaced review bring them back over the next few days.

- Explain the difference between a secant line and a tangent line, and what each slope represents.
- Write the limit definition of the derivative and say what $h \rightarrow 0$ means in plain words.
- Name the four derivative notations and identify which is Leibniz and which is Lagrange.
- Estimate $f'(2)$ for $f(x) = x^3$ using the difference quotient with $h = 0.1$, then compare with the exact value.
- State which direction the differentiability-continuity implication runs, and name the three cases where continuity holds but differentiability fails.
- Explain why $|x|$ is continuous but not differentiable at $x = 0$.
- Apply the power rule to x^5 , $\sqrt{x}$, and $1/x^2$, showing the rewrite step each time.
- State the constant, constant multiple, sum, and difference rules, then use them on $4x^3 - 2x + 7$.
- Recite the derivatives of $\sin x$, $\cos x$, e^x , and $\ln x$ from memory, including the sign on $\cos x$.
- Explain why the power rule must never be applied to e^x .
- State the product rule and apply it to $x^2 \cdot \sin x$.
- State the quotient rule and apply it to $(x^2 + 1) / (x - 1)$, simplifying the numerator.
- Recite the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$, and name which ones are negative.
- Derive $d/dx[\tan x] = \sec^2 x$ from the quotient rule in two lines.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Differentiation deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Secant line, tangent line, derivative of a function, derivative notation, difference quotient, differentiable, differentiability implies continuity, Power Rule, Constant Rule, Sum Rule, Difference Rule, Constant Multiple Rule, derivative of $\sin x$, derivative of $\cos x$, derivative of e^x , derivative of $\ln x$, Product Rule, Quotient Rule, derivative of $\tan x$, derivative of $\cot x$, derivative of $\sec x$, derivative of $\csc x$.

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