

Unit 1: Limits and Continuity

Unit 1 builds the foundation of calculus. It covers average and instantaneous rates of change, the definition and notation of limits, techniques for evaluating limits, types of discontinuities, continuity, asymptotes, and the Intermediate Value Theorem.

AP Calculus AB

Limits and Continuity

About 12 minutes to read

How to use this guide

Read it in order the first time because the topics build on each other. Rates of change motivate the limit, the limit lets you evaluate what functions approach, limits define continuity, and continuity is what the Intermediate Value Theorem needs. Exam questions often give a graph or an expression and ask you to classify what is happening at a point.

After the first read, use the trap boxes and the tables to review the distinctions that exam questions test most often. Finish with the practice questions, then complete the recall check on the last page out loud and note any items you cannot explain yet.

What this unit is worth. Limits and Continuity is about 10 to 13 percent of the AP Calculus AB exam. It also carries more weight than that number suggests, because the limit is the definition underneath the derivative in Unit 2 and the integral in Unit 6. If limits feel shaky now, differentiation will feel shaky later.

1.1 Introducing Calculus: Can Change Occur at an Instant?

Calculus starts with a question about change. The **average rate of change** of a function over an interval is the change in output divided by the change in input, $[f(b) - f(a)] / (b - a)$. On a graph it is the slope of the **secant line** through the two points $(a, f(a))$ and $(b, f(b))$. For $f(x) = x^2$, the average rate of change over $[1, 2]$ is $(4 - 1) / (2 - 1) = 3$.

The **instantaneous rate of change** is the rate at a single point. It is defined as the limit of average rates of change as the interval shrinks down to that point. On a graph it is the slope of the **tangent line**, and it is what the derivative measures. For $f(x) = x^2$ at $x = 1$, shrink the interval to $[1, 1 + h]$: the average rate is $[(1 + h)^2 - 1] / h = (2h + h^2) / h = 2 + h$, and as h approaches 0 this approaches 2. So the instantaneous rate at $x = 1$ is 2.

Trap. An average rate over an interval is not the rate at a point. The secant slope 3 above is the average over $[1, 2]$, while the instantaneous rate at $x = 1$ is 2. Questions that ask for the rate *at* a point want the limit, not the secant.

1.2 Defining Limits and Using Limit Notation

The **limit of a function**, written $\lim_{x \rightarrow c} f(x) = L$, means that $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to c , with x not equal to c . Read the notation as "the limit of f of x as x approaches c equals L ." On the exam, correct **limit notation** is required when you justify a claim about a limit, so write the $\lim_{x \rightarrow c}$ part every time.

The function does not need to be defined at c for the limit to exist. The limit describes what the function approaches, not what happens at the point itself. A limit value and a function value are two different numbers that happen to agree when a function is continuous.

Trap. The most common error in this unit is answering with $f(c)$ when the question asks for $\lim_{x \rightarrow c} f(x)$. If the function has a hole at $x = c$, the limit is still whatever the surrounding values approach.

1.3 Estimating Limit Values from Graphs

A **one-sided limit** looks at x approaching c from one side only. From the right it is written $\lim_{x \rightarrow c^+} f(x)$, and from the left $\lim_{x \rightarrow c^-} f(x)$. On a graph, trace the curve toward $x = c$ from each direction separately and read the y -value each side approaches. The two-sided limit $\lim_{x \rightarrow c} f(x)$ exists only if both one-sided limits exist and are equal to each other.

This is the tool for jump discontinuities. If the left branch approaches $y = 1$ and the right branch approaches $y = 4$ at $x = 2$, then $\lim_{x \rightarrow 2^-} f(x) = 1$ and $\lim_{x \rightarrow 2^+} f(x) = 4$, and because they differ, $\lim_{x \rightarrow 2} f(x)$ does not exist. The value $f(2)$, wherever the dot is drawn, does not change either one-sided limit.

Trap. When the two sides disagree, the two-sided limit does not exist, even if the function has a defined value at the point. A filled-in dot at $x = c$ fixes $f(c)$, not the limit.

1.5 Determining Limits Using Algebraic Properties of Limits

The **algebraic properties of limits**, also called the limit laws, let you break a limit apart. The limit of a sum, difference, product, or quotient equals the corresponding combination of the individual limits, as long as each limit exists and the denominator limit is not zero. Constant multiples and powers work the same way.

The first move is almost always direct substitution, which the limit laws justify for polynomials and for rational functions wherever the denominator is not zero. For example, $\lim_{x \rightarrow 2} (x^2 + 3x) / (x + 4) = (4 + 6) / (2 + 4) = 10/6 = 5/3$.

If substitution gives $0/0$, the expression is indeterminate and needs algebra before you can judge it. Factoring and canceling is the most common fix. For $\lim_{x \rightarrow 3} (x^2 - 9) / (x - 3)$, substitution gives $0/0$, but factoring gives $(x - 3)(x + 3) / (x - 3)$, which equals $x + 3$ for $x \neq 3$, so the limit is 6.

WHAT SUBSTITUTION GIVES	WHAT TO TRY
A number	Done. That is the limit.
$0/0$	Factor and cancel, or use another algebraic rewrite, then substitute again.
A nonzero number over 0	The function grows without bound. Check each side for an infinite limit and a vertical asymptote.
An oscillating factor like $\sin(1/x)$	Try the Squeeze Theorem from topic 1.8.

Trap. The form $0/0$ is indeterminate, not an answer. It means the algebra is hiding the real value, as in the $(x^2 - 9) / (x - 3)$ example where the limit is 6. Never stop at $0/0$.

1.8 Determining Limits Using the Squeeze Theorem

The **Squeeze Theorem** handles functions you cannot evaluate directly. If $g(x) \leq f(x) \leq h(x)$ for x near c , and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$. The outer functions pin the middle one down to the same limit.

The standard example is $\lim_{x \rightarrow 0} x \cdot \sin(1/x)$. The sine factor oscillates between -1 and 1 forever, so direct substitution fails, but $-|x| \leq x \cdot \sin(1/x) \leq |x|$ for all x , and both bounds approach 0 as $x \rightarrow 0$. The squeeze forces the limit to 0 . A related result worth memorizing is $\lim_{x \rightarrow 0} \sin x / x = 1$, which is proved the same way with a geometric squeeze and appears constantly in later units.

Trap. The Squeeze Theorem needs both bounding functions to approach the *same* limit. If the bounds approach different values, the theorem says nothing, and the middle function might do anything.

1.10 Exploring Types of Discontinuities

A discontinuity is a break in the graph, and there are three types. A **removable discontinuity** is a hole: the limit exists at $x = c$, but $f(c)$ is undefined or does not equal the limit. Filling in the hole at the limit value would repair it. A **jump discontinuity** is where the left- and right-hand limits exist but are not equal, so the graph jumps from one value to another. A **discontinuity due to a vertical asymptote** is where the function grows without bound as x approaches c , an infinite limit.

TYPE	WHAT YOU SEE	EXAMPLE
Removable	A hole. The limit exists but the point is missing or misplaced.	$f(x) = (x^2 - 9) / (x - 3)$ at $x = 3$. The limit is 6 , but $f(3)$ is undefined.
Jump	A step. The one-sided limits exist and disagree.	A piecewise function with $\lim_{x \rightarrow 0^-} = 1$ and $\lim_{x \rightarrow 0^+} = 3$.
Vertical asymptote	The graph shoots up or down without bound near $x = c$.	$f(x) = 1 / (x - 2)$ at $x = 2$.

Trap. A hole and a vertical asymptote are not the same break. If the limit exists at the point, the discontinuity is removable. If the function grows without bound, it is an infinite discontinuity. Check the limit first, then classify.

1.11 Defining Continuity at a Point

Continuity at a point has a three-part definition. A function f is continuous at $x = c$ when $f(c)$ is defined, $\lim_{x \rightarrow c} f(x)$ exists, and the two are equal: $\lim_{x \rightarrow c} f(x) = f(c)$. In words, you can draw through the point without lifting the pencil, and the value you approach matches the value that is there.

Each discontinuity type fails the definition in a characteristic way. A removable discontinuity passes the limit condition but fails the equality, because $f(c)$ is missing or wrong. A jump discontinuity fails the limit condition itself, because the two sides disagree. An infinite discontinuity fails both, since the function is undefined at c and the limit does not exist as a real number.

Trap. Justification questions want all three conditions named. Saying "the function is defined at $x = 2$ " is not enough. State that $f(2)$ is defined, the limit exists, and the limit equals $f(2)$, with the values shown.

1.12 Confirming Continuity over an Interval

Continuity over an interval means the function is continuous at every point of the interval. For interior points that is the three-condition test from 1.11. At an endpoint of a closed interval, only the relevant one-sided limit needs to equal the function value, since there is no other side to check.

This is why the familiar function families are easy to work with. Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined, which is everywhere except where the denominator is zero. Sums, differences, products, and quotients of continuous functions stay continuous wherever the result is defined, so most expressions on the exam are continuous on their domains by construction.

Trap. "Continuous on its domain" is not the same as "continuous everywhere." The function $f(x) = 1/x$ is continuous on its domain, but it is not continuous at $x = 0$, because 0 is not in its domain. Read which claim the question is actually making.

1.14 Connecting Infinite Limits and Vertical Asymptotes

An **infinite limit** describes a function growing without bound near a point, written $\lim_{x \rightarrow c} f(x) = \infty$ or $-\infty$. Strictly speaking the limit does not exist as a real number, but the infinity notation records the behavior, and the exam expects you to use it. For $f(x) = 1 / (x - 2)^2$, the values blow up to positive infinity from both sides, so $\lim_{x \rightarrow 2} f(x) = \infty$.

A **vertical asymptote** is the line $x = c$, and it appears exactly when f has an infinite limit as x approaches c from at least one side. One side is enough. The sign can differ by side, so check each. For $f(x) = 1/x$, $\lim_{x \rightarrow 0^+} 1/x = \infty$ while $\lim_{x \rightarrow 0^-} 1/x = -\infty$, and $x = 0$ is a vertical asymptote either way.

QUESTION BEING ASKED	WHAT YOU ARE FINDING
$\lim_{x \rightarrow c} f(x)$ with x finite	A value, a one-sided split, or an infinite limit at a point. May give a vertical asymptote.
$\lim_{x \rightarrow \infty} f(x)$ with x unbounded	End behavior. May give a horizontal asymptote. See topic 1.15.

Trap. Infinite limits and limits at infinity are different ideas that the notation makes look similar. The first is about x approaching a finite number with the function blowing up. The second is about x itself growing without bound. Match the question to the right row of the table before you start.

1.15 Connecting Limits at Infinity and Horizontal Asymptotes

A **limit at infinity**, $\lim_{x \rightarrow \infty} f(x) = L$, means $f(x)$ approaches L as x grows without bound. A **horizontal asymptote** is the line $y = L$, and it appears when $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$. **End behavior** is the description of what happens in both directions, $x \rightarrow \infty$ and $x \rightarrow -\infty$, often summarized by the horizontal asymptotes.

For rational functions, compare degrees. If the degree on top is smaller, the limit is 0. If the degrees are equal, divide by the highest power and read the ratio of leading coefficients: $\lim_{x \rightarrow \infty} (3x^2 + 1) / (2x^2 - 5) = \lim_{x \rightarrow \infty} (3 + 1/x^2) / (2 - 5/x^2) = 3/2$, so $y = 3/2$ is a horizontal asymptote. If the degree on top is larger, the function grows without bound and there is no horizontal asymptote.

Trap. A graph is allowed to cross its horizontal asymptote. The asymptote only describes where the function settles as x gets very large. A question that asks whether the graph crosses $y = 3/2$ at some finite x is testing something the asymptote cannot answer.

1.16 Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** says that a continuous function cannot skip values. If f is continuous on the closed interval $[a, b]$, and N is any value between $f(a)$ and $f(b)$, then there is at least one c in the open interval (a, b) with $f(c) = N$. The hypotheses matter: the interval must be closed, the function must be continuous on it, and N must lie between the endpoint values.

A typical justification: $f(x) = x^3 - x - 2$ is a polynomial, so it is continuous everywhere, in particular on $[1, 2]$. Since $f(1) = 1 - 1 - 2 = -2$ and $f(2) = 8 - 2 - 2 = 4$, and 0 lies between -2 and 4 , the IVT guarantees some c in $(1, 2)$ with $f(c) = 0$. The theorem promises existence, not the location of c .

Trap. The IVT needs all of its hypotheses. If the function is not continuous on $[a, b]$, or if N is not between $f(a)$ and $f(b)$, the conclusion does not follow. Also, the converse is false: a discontinuous function can still hit every intermediate value, but the IVT is not what guarantees it.

Confusions That Cost Points

PAIR	HOW TO KEEP THEM STRAIGHT
$\lim_{x \rightarrow c} f(x)$ vs $f(c)$	The limit is what the function approaches. $f(c)$ is what is actually there. They agree only when f is continuous at c .
One-sided vs two-sided limit	The two-sided limit exists only when both one-sided limits exist and agree. A jump kills the two-sided limit.
Removable vs jump discontinuity	Removable means the limit exists and the point is the problem. Jump means the limit itself does not exist.
Infinite limit vs limit at infinity	Infinite limit: x approaches a finite c and f blows up, giving a vertical asymptote. Limit at infinity: x grows without bound, giving end behavior and possibly a horizontal asymptote.
IVT hypotheses vs conclusion	Continuity on $[a, b]$ plus N between $f(a)$ and $f(b)$ gives existence of c . Drop a hypothesis and the guarantee is gone.

Practice Questions

Original questions written for this guide in the style of the AP exam. Answers and explanations are on the next page, so complete the questions before checking them.

1. $\lim_{x \rightarrow 3} (x^2 - 9) / (x - 3) =$

- A. 0
- B. 6
- C. Does not exist
- D. Undefined

2. The graph of f has a jump at $x = 2$. The left branch approaches $y = 1$, the right branch approaches $y = 4$, and $f(2) = 4$. What is $\lim_{x \rightarrow 2} f(x)$?

- A. 1
- B. 4
- C. Does not exist
- D. 2

3. Let $f(x) = (x^2 - 4) / (x - 2)$ for $x \neq 2$, and $f(2) = 5$. Which statement is true?

- A. f is continuous at $x = 2$
- B. f has a removable discontinuity at $x = 2$
- C. f has a jump discontinuity at $x = 2$
- D. $\lim_{x \rightarrow 2} f(x) = 5$

4. If $-x^2 \leq g(x) \leq x^2$ for all x near 0, then $\lim_{x \rightarrow 0} g(x) =$

- A. 0
- B. Does not exist
- C. 1
- D. Cannot be determined from this information

5. $\lim_{x \rightarrow 2^-} 1 / (x - 2) =$

- A. ∞
- B. $-\infty$
- C. 0
- D. Does not exist

6. $\lim_{x \rightarrow \infty} (4x^3 - x) / (2x^3 + 5) =$

- A. 2
- B. 0
- C. ∞
- D. $-1/5$

7. The function f is continuous on $[0, 3]$, with $f(0) = -1$ and $f(3) = 5$. Which of the following must be true?

- A. $f(1.5) = 2$
- B. There is a value c in $(0, 3)$ such that $f(c) = 0$
- C. f has a zero at $x = 1.5$
- D. f is differentiable on $(0, 3)$

8. The graph of f has a horizontal asymptote $y = 3$ as $x \rightarrow \infty$. Which of the following must be true?

- A. $\lim_{x \rightarrow \infty} f(x) = 3$
- B. $f(3) = 3$
- C. The graph of f never crosses the line $y = 3$
- D. $\lim_{x \rightarrow 3} f(x) = \infty$

Answer Key

- 1. B.** Direct substitution gives $0/0$, so factor: $(x - 3)(x + 3) / (x - 3) = x + 3$ for $x \neq 3$, and the limit is 6. A stops at the indeterminate form $0/0$ and treats it as an answer. C confuses "indeterminate" with "does not exist"; the algebra reveals a finite limit. D is vague hand-waving that avoids doing the algebra.
- 2. C.** The one-sided limits are 1 from the left and 4 from the right. They disagree, so the two-sided limit does not exist. A reports only the left-hand limit. B reports the function value $f(2)$, which equals the right-hand limit here but is not the two-sided limit. D confuses the x -value where the jump happens with a y -value.
- 3. B.** Factoring gives $(x - 2)(x + 2) / (x - 2) = x + 2$ for $x \neq 2$, so the limit is 4, but $f(2) = 5$. The limit exists while the function value disagrees, which is the definition of a removable discontinuity. A ignores the mismatch between the limit and $f(2)$. C is wrong because a jump requires disagreeing one-sided limits, and here both sides approach 4. D confuses $f(2) = 5$ with the limit, which is 4.
- 4. A.** Both bounds approach 0 as $x \rightarrow 0$: $\lim_{x \rightarrow 0} x^2 = 0$ and $\lim_{x \rightarrow 0} (-x^2) = 0$. The Squeeze Theorem forces $g(x)$ to 0 as well. B ignores the theorem entirely. C may come from misreading x^2 near 0 as approaching 1. D is the trap of thinking the bounds are not enough; matching bounds are exactly what the theorem needs.
- 5. B.** From the left, $x - 2$ is a small negative number, so $1 / (x - 2)$ is a large negative number: the limit is $-\infty$. A ignores the side and reports the right-hand behavior instead. C confuses an infinite limit with a limit at infinity of $1/x$. D is tempting because infinite limits do not exist as real numbers, but the question asks for the one-sided behavior, which is precisely $-\infty$.
- 6. A.** The degrees are equal, so divide by x^3 : $(4 - 1/x^2) / (2 + 5/x^3)$, which approaches $4/2 = 2$. B would be correct only if the denominator degree were larger. C treats the leading term as dominating without checking that the degrees match. D takes the ratio of the constant terms, which is meaningless here since the x^3 terms control the behavior.
- 7. B.** The IVT applies: f is continuous on $[0, 3]$ and 0 lies between $f(0) = -1$ and $f(3) = 5$, so some c in $(0, 3)$ satisfies $f(c) = 0$. A and C both claim the theorem locates the zero; it only guarantees existence, never the position. D confuses continuity with differentiability, which the IVT never requires.
- 8. A.** A horizontal asymptote $y = 3$ as $x \rightarrow \infty$ is exactly the statement that $\lim_{x \rightarrow \infty} f(x) = 3$. B confuses the asymptote's y -value with a function value at $x = 3$. C is the classic trap: a graph may cross its horizontal asymptote at finite x values; the asymptote only governs far-out behavior. D confuses a horizontal asymptote with a vertical one.

When you check your answers, note which distinction each miss came from. Make a flashcard for that distinction and drill it spaced out over the next few days instead of rereading the whole section. If you missed one of these questions, the same distinction is worth practicing again in Rycal, where the Limits and Continuity deck has flashcards for it and more practice questions use the same kinds of traps.

One-Page Recall Check

Say each answer out loud before you look back, and mark the ones you cannot finish. Anything you cannot say out loud yet belongs in your flashcard deck. In Rycal, add those items to the Limits and Continuity deck and let spaced review bring them back over the next few days.

- State the average rate of change formula and explain what the secant line represents.
- Explain how the instantaneous rate of change is defined as a limit of average rates.
- Write $\lim_{x \rightarrow c} f(x) = L$ in words, and explain why $f(c)$ need not exist.
- State the condition for a two-sided limit to exist in terms of one-sided limits.
- List the limit laws and the two conditions needed to use the quotient law.
- Evaluate $\lim_{x \rightarrow 3} (x^2 - 9) / (x - 3)$ from scratch, showing the algebra.
- State the Squeeze Theorem and use it to find $\lim_{x \rightarrow 0} x \cdot \sin(1/x)$.
- Describe each discontinuity type and name which continuity condition it violates.
- State the three conditions for continuity at a point.
- Explain what changes at an endpoint when checking continuity over a closed interval.
- Explain how a vertical asymptote follows from an infinite limit on at least one side.
- Give the degree rules for limits at infinity of rational functions.
- State the Intermediate Value Theorem with all of its hypotheses.

Where to go next. Turn every missed item above into flashcards and drill them spaced out over several days rather than in one sitting. In Rycal, open the Limits and Continuity deck under AP Calculus AB. The deck covers the terms in this guide, and its practice questions target the same traps named here. If you have a test date, add it in the Test Planner. You can also start your next review with a Brain Dump, then check what you missed against this guide.

Key terms for this unit

Average rate of change, Instantaneous rate of change, Limit of a function, Limit notation, One-sided limit, Algebraic properties of limits (limit laws), Squeeze Theorem, Removable discontinuity, Jump discontinuity, Discontinuity due to a vertical asymptote, Continuity at a point, Continuity over an interval, Infinite limit, Vertical asymptote, Limit at infinity, Horizontal asymptote, End behavior, Intermediate Value Theorem (IVT).

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